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MACMILLAN AND CO. LONDON.

ELEMENTARY ALGEBRA

BY

CHARLES SMITH M.A.

MASTER OF SIDNEY SUSSEX COLLEGE, CAMBRIDGE.

Toronto:
THE COPP CLARK CO. LIMITED.

London:
MACMILLAN AND CO.
AND NEW YORK.

1894

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PREFACE.

The second edition of this work

THE second edition of this book differs from the first in some important particulars. The whole book has been thoroughly revised and the early chapters remodelled and simplified; the number of examples has been very greatly increased; and chapters on Logarithms and Scales of Notation have been added. It is hoped that the changes which have been made will increase the usefulness of the work. The book is now stereotyped and only corrections and slight alterations will be made in any future editions.

I am indebted to several of my friends for their kindness in looking over the proof sheets and in working the examples. My especial thanks are due to Mr J. Carpenter, M.A. and to Mr A. G. Cracknell, B.A. St Ca

CHARLES SMITH.

SIDNEY SUSSEX COLLEGE,
April, 1890.

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ALGEBRA.

CHAPTER I.

DEFINITIONS.

1. ALGEBRA is a science which treats of the relations of numbers.

In Arithmetic numbers are represented by *figures*, each of which has one and only one meaning.

In Algebra numbers are represented either by figures or by the **letters of the alphabet.**

> It is proved in Arithmetic that when any one number is multiplied by any second number the result is the same as when the second number is multiplied by the first. If now we represent the two numbers by the two letters a and b respectively, the above property of numbers is expressed by $a \times b = b \times a$; and the gain in conciseness of statement by the use of letters is apparent.

In Algebra each letter may usually be supposed to represent any number whatever; but, in any connected series of operations, each letter must throughout be supposed to represent the same number.

2. The signs which are used in Arithmetic are also used in Algebra; but, as we shall see later on, it is necessary to extend the meaning of some of the signs.

3. The sign +, which is read **'plus,'** is placed before a number, or a number represented by a letter, to indicate that it is to be *added* to what has gone before.

> Thus $6+3$ means that 3 is to be added to 6; $6+3+2$ means that 3 is to be added to 6 and then 2 added to the result: so also $a+b$ means that the number which is represented by b is to be added to the number which is represented by a; or, expressed more briefly, it means that b is to be added to a.

4. The sign −, which is read **'minus,'** is placed before a number to indicate that it is to be *subtracted* from what has gone before.

> Thus $6-3$ means that 3 is to be subtracted from 6, $a-b$ means that b is to be subtracted from a, and $a-b+c$ means that b is to be subtracted from a and then c added to the result.

It should be noticed that in a series of *additions and subtractions the order of the operations is from left to right.*

5. The sign of *multiplication* is ×, which is read **'multiplied by'** or **'into.'**

> Thus 6×3 means that 6 is to be multiplied by 3, $a\times b$ means that a is to be multiplied by b, and $a\times b\times c$ means that a is to be multiplied by b and then the result multiplied by c.

The sign of multiplication is generally omitted between two letters, or between a number and a letter, and the letters are simply placed side by side. Sometimes the × is replaced by a point.

> Thus ab or $a\,.\,b$ means the same as $a\times b$, and $2abc$ or $2a\,.\,b\,.\,c$ the same as $2\times a\times b\times c$.

6. The sign of *division* is ÷, which is read **'divided by'** or **'by.'**

> Thus $6\div 3$ means that 6 is to be divided by 3, $a\div b$ means that a is to be divided by b, and $a\div b\div c$ means that a is to be divided by b and then the result divided by c; also $a\div b\times c$ means that a is to be divided by b and then the result multiplied by c.

The operation of division is often indicated by placing the dividend over the divisor with a line between them: thus $\frac{a}{b}$ is used instead of $a \div b$.

It should be noticed that in a series of *multiplications and divisions the order of the operations is from left to right.*

7. When two or more numbers are multiplied together the result is called the **continued product**, or simply the **product**; and each number is called a **factor** of the product.

8. When the factors of a product are considered as divided into two sets, each is called the **coefficient**, that is the *co-factor* of the other.

Thus in $3abx$, 3 is the coefficient of abx; also $3a$ is the coefficient of bx, and $3ab$ is the coefficient of x.

When one of the factors of a product is a number expressed in figures, it is called the **numerical coefficient** of the other factors.

EXAMPLES. I.

Calculate the values of

1. $7+6+4$. 2. $5-3+4$. 3. $11+7-12-6$.
4. $7\times6\times4$. 5. $5\div3\times4$. 6. $11\times7\div12\div6$.

If $a=1$, $b=2$, $c=3$ and $d=4$, find the numerical values of

7. $c-b$. 8. $d-a$. 9. $7a-3b$. 10. $10b-6c$.
11. $5a-2b+6c-4d$. 12. $13a-6b+7c-5d$.
13. $18b-3c-4d+9a$. 14. $20ab-3cd$.
15. $4da-2bc$. 16. $abc+bcd+cda+dab$.

If $a=6$, $b=2$, $c=5$ and $d=0$, find the values of

17. $3ac+2bc+ca$. 18. $7ad+9bc-3ca$. 19. $a\times c\div b$.
20. $a\div c\times b$. 21. $2c\div a\div b$.

22. What are the coefficients of x in $3x$, $4bx$, $5bcx$ and $16abcx$?

23. What are the coefficients of xy in $4xy$, $5axy$, $7abxy$ and $19abcxy$? What are the numerical coefficients?

9. When a product consists of the same factor repeated any number of times it is called a **power** of that factor. Thus aa is called the *second power* of a, aaa is called the *third power* of a, $aaaa$ is called the *fourth power* of a, and so on. Sometimes a is called the *first power* of a.

Special names are also given to aa and aaa; they are called respectively the **square** and the **cube** of a.

10. Instead of writing aa, aaa etc., a more convenient notation is adopted as follows: a^2 is used instead of aa, a^3 is used instead of aaa, and a^n is used instead of $aaa\ldots\ldots$, the factor a being taken n times; the small figure, or letter, placed above and to the right of a, shewing the number of times the factor a is to be taken. So also a^3b^2 is written instead of $aaabb$, and similarly in other cases.

The small figure, or letter, placed above a symbol to indicate the number of times that symbol is to be taken as a factor is called the **index** or the **exponent.** Thus a^n means that the factor a is to be taken n times, or that the nth *power* of a is to be taken, and n is called the *index.*

When the factor a is only to be taken once, we do not write it a^1, but simply a.

11. The quantity which when squared is equal to any number a is called the **square root** of a, and is represented by the symbol $\sqrt[2]{a}$, or more often by $\sqrt{a}$: thus 2 is $\sqrt{4}$, for 2^2 is 4.

The quantity which when cubed is equal to any number a is called the **cube root** of a, and is represented by the symbol $\sqrt[3]{a}$: thus $3 = \sqrt[3]{27}$, for $3^3 = 27$.

In general, the quantity which when raised to the nth power, where n is any whole number, is equal to any number a, is called the nth root of a, and is represented by the symbol $\sqrt[n]{a}$.

The sign $\sqrt{}$ was originally the initial letter of the word *radix*. It is often called the **radical** sign.

A root which cannot be obtained exactly is called a **surd**, or an **irrational** quantity: thus $\sqrt{7}$ and $\sqrt[3]{4}$ are surds.

The approximate value of a surd, for example of $\sqrt{7}$, can be found, to any degree of accuracy which may be desired, by the ordinary arithmetical process; but we are not required to find these approximate values in Algebra: for us $\sqrt{7}$, is simply that quantity which when squared will become 7.

12. The sign $=$, which is read '**equals**,' or '**is equal to**,' is placed between two quantities to indicate that they are *equal* to one another.

Thus $5+7=12$, which is read five plus seven equals twelve.

The sign $>$ indicates that the number which precedes the sign **is greater than** that which follows it.

Thus $a > b$ means that a is greater than b.

The sign $<$ indicates that the number which precedes the sign is **less than** that which follows it.

Thus $a < b$ means that a is less than b.

The sign $\because$ is written for the word **because** or **since**.

The sign $\therefore$ is written for the word **therefore** or **hence**.

13. A collection of algebraical symbols, that is of letters, figures, and signs, is called an **algebraical expression**.

The parts of an algebraical expression which are connected by the signs $+$ or $-$ are called the **terms**.

Thus $2a-3bx+5cy^2$ is an algebraical expression containing the three terms $2a$, $-3bx$ and $+5cy^2$.

14. When two terms contain the same letters every one of which is raised to the same power in both, they are called **like terms**.

Thus $3ab^2x^3$ and $7ab^2x^3$ are *like* terms; but, although $3a^2bx^3$ and $7a^2b^2x^3$ contain the same letters, they are not like terms, for all the letters are not raised to the same power.

15. A **monomial expression** is one which contains only one term, and a **multinomial** expression is one which contains more than one term.

Thus $5ab^2cx$ is a monomial expression, and $a + b$ is a multinomial expression.

An expression which consists of *two* terms is often called a **binomial** expression, and an expression which consists of *three* terms is called a **trinomial** expression.

Monomial and multinomial expressions are sometimes called respectively **simple** and **compound** expressions.

16. **Brackets.** To denote that an algebraical expression is to be treated *as a whole*, it is put between brackets, and all operations denoted by signs within the brackets are to be performed before any operation denoted by a sign outside the brackets. The various shapes of brackets are ( ), { }, [ ].

Thus $(a+b)\,c$ means that b is to be added to a and that the result is to be multiplied by c; again $(a+b)^3$ means that b is to be added to a and that the cube of the result is to be formed; also $(a+2b)\,(c-3d)$ means that $2b$ is to be added to a and that $3d$ is to be taken from c, and that the first of these results is then to be multiplied by the second.

Instead of a pair of brackets a line called a **vinculum** is often drawn over the expression which is to be treated as a whole: thus $a-\overline{b-c}$ is equivalent to $a-(b-c)$, and $\sqrt{a+b}$ is equivalent to $\sqrt{}(a+b)$.

When no vinculum or bracket is used, a radical sign refers only to the number or letter which immediately follows it: thus $\sqrt{}2a$ means that the square root of 2 is to be multiplied by a, whereas $\sqrt{2a}$ means the square root of $2a$; also $\sqrt{}a+x$ means that x is to be added to the square root of a, but $\sqrt{a+x}$ means the square root of the sum of a and x.

The line between the numerator and denominator of a fraction acts as a vinculum, for $\frac{a+b}{12}$ is the same as $\frac{1}{12}(a+b)$.

Note. It should be carefully noticed that every *term* of an algebraical expression must be added or subtracted *as a whole*, as if it were enclosed in brackets. Thus, in the expression $a+bc-d\div e+f$, b must be multiplied by c before addition, and d must be divided by e before subtraction, just as if the expression were written $a+(bc)-(d\div e)+f$.

Ex. Find the values of

(i) $2a-bc+cd-6b\div a+\frac{2c}{a}$, (ii) $(a+b)^3(2b-3c)^2$,

(iii) $a^b+b^c+c^a$, (iv) $\sqrt[3]{}\{7a^3+(b+c)^3+d^3\}$,

when $a=4$, $b=3$, $c=1$, $d=0$.

(i) $2a-bc+cd-6b\div a+\frac{2c}{a}$

$$=2\times 4-3\times 1+1\times 0-6\times 3\div 4+\frac{2\times 1}{4}$$
$$=8-3+0-\tfrac{9}{2}+\tfrac{1}{2}=1.$$

(ii) $(a+b)^3(2b-3c)^2=(4+3)^3(2\times 3-3\times 1)^2=7^3\times 3^2$
$$=343\times 9=3087.$$

(iii) $a^b+b^c+c^a=4^3+3^1+1^4=64+3+1=68.$

(iv) $\sqrt[3]{}\{7a^3+(b+c)^3+d^3\}=\sqrt[3]{}\{7\times 4^3+4^3+0^3\}$
$$=\sqrt[3]{}\{7\times 64+64+0\}=\sqrt[3]{}(448+64)=\sqrt[3]{512}=8.$$

EXAMPLES. II.

1. Write down the values of 2^4, 3^3, 4^3, 4^4, $\sqrt{64}$, $\sqrt[3]{64}$, $\sqrt[4]{16}$, $\sqrt[3]{125}$, $\sqrt[4]{625}$ and $\sqrt[5]{32}$.

If $a=2$, $b=3$, $c=4$ and $d=5$, find the numerical values of

2. a^2+b^2. **3.** c^2+d^2. **4.** $6a^2-2b^2$.

5. $4b^2-c^2+5d^2$. **6.** $a^2b^2+c^2d^2$. **7.** $ab^2c^2-a^2bc$.

2 2 2

If $a=2$, $b=3$, $c=4$ and $d=5$, find the numerical values of

8. $\frac{d^3}{5}-\frac{c^3}{3}+\frac{a^2b^2}{27}$. **9.** $\frac{1}{9}bc+\frac{1}{8}ca+\frac{1}{7}ab$. **10.** $10c^3d^3-16a^2b^2$.

11. $\frac{1}{3}a^2b^2c^3-\frac{1}{9}ab^2c^2d$. **12.** $\frac{ab^3c^4}{16}-\frac{a^4bc^2d}{20}$.

If $a=5$, $b=3$, $c=1$, $d=0$, find the numerical values of

13. $(2a+5b)(3b-6c)$. **14.** $(a+2b)(c+2d)$.

15. $(3a-4b)^3-2(3b-6c)^2+2(ad+bc)^2$.

16. $4a^3+4b^3+4c^3-3(b+c)(c+a)(a+b)$.

17. $5(a+c)^3(b-c)^2-\frac{1}{25}(a-2d)^3(b+3c)^2$.

18. Shew that x^2-5x+6 is equal to zero, if $x=2$ or if $x=3$.

19. Shew that $2x^3-11x^2+17x-6$ is equal to zero, if $x=2$, or if $x=3$, or if $x=\frac{1}{2}$.

If $a=5$, $b=4$, $c=\frac{1}{2}$, find the numerical values of

20. $\sqrt{a^2-b^2}$. **21.** $\sqrt{5a}$. **22.** $\sqrt{2bc}+3a$. **23.** $\sqrt[3]{bc+a^2}$.

24. $\sqrt[3]{(2a^2+b^2-8c^2)}$. **25.** $\sqrt[3]{(4a^2-\frac{1}{2}b^2+\frac{1}{2}c^2-1)}$.

26. $\sqrt{(a+b)}\sqrt[3]{(3ab+2bc)}$.

27. Find the numerical value of

$$(a+b)^2(x+y)^2-4(ax+by)(bx+ay),$$

if $x=5$, $y=8$, $a=6$ and $b=4$.

28. Find the value of $\sqrt{s(s-a)(s-b)(s-c)}$ when $a=9$, $b=12$, $c=15$ and $s=18$.

29. Find the value of $\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$ when $a=9$, $b=12$, $c=15$, and $2s=a+b+c$.

30. Find, when $a=8$, $b=5$, $c=3$, the numerical value of

$$\sqrt{\{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4\}}.$$

31. Verify that a^2-b^2 and $(a+b)(a-b)$ are equal to one another (i) when $a=6$, $b=3$; (ii) when $a=9$, $b=4$; and (iii) when $a=12$, $b=7$.

32. Verify that the expressions

$$a^3-b^3,\ (a-b)(a^2+ab+b^2),\ (a-b)^3+3ab(a-b),$$

and

$$(a+b)^3-3ab(a+b)-2b^3$$

are all equal to one another (i) when $a=3$, $b=2$; (ii) when $a=6$, $b=3$; and (iii) when $a=5$, $b=2$.

CHAPTER II.

POSITIVE AND NEGATIVE QUANTITIES. ADDITION. SUBTRACTION. BRACKETS.

17. ALL concrete quantities must be measured by the number of times each contains some unit of its own kind. Now a sum of money may be either a *receipt* or a *payment*, it may be either a *gain* or a *loss*; motion along a straight line may be in either of *two opposite* directions; a period of time may be either *before* or *after* some particular epoch; and so in very many other cases. Thus many concrete magnitudes are capable of existing in two diametrically opposite states.

18. Now whatever kind of quantity we are considering + 4 will stand for what *increases* that quantity by 4 units, and – 4 will stand for whatever *decreases* the quantity by 4 units.

If we are calculating the amount of a man's property (estimated in pounds), + 4 will stand for what increases his property by £4, that is + 4 will stand for £4 that he possesses, or that is owing to him; so also – 4 will stand for whatever decreases his property by £4, that is, – 4 will stand for £4 that he owes. If, on the other hand, we are calculating the amount of a man's debts, + 4 will stand for whatever increases his debts, that is + 4 will now stand for a debt for £4; so also -4 will now stand for whatever decreases his debts, that is -4 will stand for £4 that he has, or that is owing to him.

If we are considering the amount of a man's gains, $+4$ will stand for what increases his total gain, that is $+4$ will stand for a *gain* of 4; so also -4 will stand for what decreases his total gain, that is -4 will stand for a loss of 4. If however we are calculating the amount of a man's losses, $+4$ will stand for a *loss* of 4, and -4 will stand for a *gain* of 4.

Again, if the magnitude to be increased or diminished is the distance from any particular point, measured in any particular direction, $+4$ will stand for a distance of 4 units in that direction, and -4 will stand for a distance of 4 units in the *opposite* direction.

19. From the above examples it will be seen that the signs $+$ and $-$ will serve to distinguish between magnitudes of opposite kinds. Thus whatever $+4$ may represent, -4 will represent an equal magnitude but of the opposite kind. The signs $+$ and $-$ are therefore used in Algebra with two entirely different meanings. In addition to their original meanings as signs of the *operations* of addition and subtraction respectively, they are also used as *marks of distinction* between magnitudes of opposite kinds.

20. A quantity to which the sign $+$ is prefixed is called a **positive quantity**, and a quantity to which the sign $-$ is prefixed is called a **negative quantity**.

The signs $+$ and $-$ are called respectively the *positive* and *negative* signs.

21. The signs $+$ and $-$ are often called **signs of affection** when they are used to indicate a *quality* of the quantities before whose symbols they are placed.

The sign +, as a sign of affection, is frequently omitted; and when neither the + nor the – sign is prefixed to a term the + sign is to be understood.

Note. Although there are many signs used in algebra, the name *sign* is often used to denote the two signs + and – exclusively. When *the sign* of a quantity is spoken of, it means the + or – sign which is prefixed to it; and when we are directed to *change the signs* of an expression, it means that we are to change the + or – before every term into – and + respectively.

22. The magnitude of a quantity considered independently of its quality, or of its sign, is called its **absolute magnitude.**

Thus a rise of 4 feet and a fall of 4 feet are equal in absolute magnitude; so also $+4$ and -4 are equal in absolute magnitude, whatever the unit may be.

ADDITION.

23. The process of finding the result when two or more quantities are taken together is called **addition**, and the result is called the **sum.**

Since a positive quantity produces an increase, and a negative quantity produces a decrease, to add a positive quantity we must *add* its absolute magnitude, and to add a negative quantity we must *subtract* its absolute magnitude.

Thus, when we add $+4$ to $+6$, we get $+6+4=+10$; and when we add -4 to $+10$, we get $+10-4=+6$.

So also, when we add $+b$ to a, we get $a+b$; and when we add $-b$ to a, we get $a-b$.

Hence $a+(+b)=a+b$ and $a+(-b)=a-b$.

We therefore have the following rule for the addition of any term:—*To add any term affix it to the expression to which it is to be added, with its sign unchanged.*

Ex. A boy played two games; in the first game he won 6 points, and in the second he won -4 points (that is he lost 4). How many did he win altogether?

The total gain in the two games together is what is meant in Algebra by the *sum* of the gains.

To obtain the total gain we must add -4 to 6, and this operation is indicated by $6+(-4)$, which by the above is $6-4=2$.

When numerical values are given to a and to b, the numerical values of $a+b$ and $a-b$ can be found; but until we know what numbers a and b represent, we cannot take any further step, and the process is considered to be algebraically complete.

24. It should be noticed that when b is greater than a, the arithmetical operation denoted by $a-b$ is impossible; for we cannot take any number from a smaller number.

Thus, if $a=3$ and $b=5$, $a-b$ will be $3-5$, and we cannot take 5 from 3. But to subtract 5 is the same as to subtract 3 and 2 in succession, so that $3-5=3-3-2=-2$: we then consider that -2 is 2 which is to be subtracted from some other algebraical expression, or that -2 is two units of the kind opposite to that represented by 2. And if -2 is a *final* result, the latter is the only view that can be taken.

In some particular cases the quantities may be such that a negative result is without meaning; for instance, if we have to find the population of a town from certain given conditions; in this case the occurrence of a negative result would shew that the given conditions could not be satisfied, but so also in this case would the occurrence of a fractional result.

EXAMPLES. III.

Find the sum of

1. 4 and -3.

2. 3 and -2.

3. 6 and -3.

4. 7 and -8.

5. 3 and -11.

6. -3 and -9.

7. 6, -2 and 7.

8. -3, -2 and 5.

9. $2a$ and $-3b$.

10. $-3a$ and $-2b$.

11. $5a$, $-6b$ and $-2c$.

12. $-3a$, $-4b$ and $7c$.

25. It follows at once from the nature of addition that the sum of two or more algebraical quantities, whether positive or negative, is the same *in whatever order the quantities may be taken.*

> For example, to find how much a man is worth, we can take into account the different items of property (debts being considered as negative) in any order we please.

It also follows that to add any algebraical expression *as a whole* gives the same result as to *add its terms separately.* But to add any term we have only to write it down, *with its sign unchanged,* after the expression to which it is to be added.

We have therefore the following

Rule :—*To add two or more algebraical expressions, write down all the terms in succession with their signs unchanged.*

> For example, the sum of $a+b$ and $c-d$ is $a+b+c-d$; also the sum of $a-b+c$ and $-d+e-f$ is $a-b+c-d+e+f$.

26. If some of the terms which are to be added are 'like' terms, we can and must collect all such terms together before the process of addition is considered to be complete. Of this we have three cases, as follows :

I. The sum of 'like' terms which have the same sign is a 'like' term which has the same sign, and whose coefficient is the arithmetical sum of their numerical coefficients.

> For example, to add $2a$ and $5a$ in succession gives the same result, whatever a may be, as to add $7a$; that is $+2a+5a=+7a$. Also to subtract $2ab$ and $5ab$ in succession gives the same result as to subtract $7ab$; that is, $-2ab-5ab=-7ab$.

II. The sum of two 'like' terms whose signs are different is a 'like' term whose coefficient is the arithmetical difference of their numerical coefficients and whose sign is that of the greater.

For example, $+5a-3a=+2a+3a-3a=+2a$; also
$$+3ab-5ab=+3ab-3ab-2ab=-2ab.$$

III. If there are several 'like' terms some of which are positive and some negative, the positive terms can be collected into one sum by I, as also the negative terms: the final sum is then obtained by II.

Thus any number of 'like' terms can be reduced to one term.

Ex. 1. Add $2a+3b$ and $a-5b$.

The sum is $2a+3b+a-5b$; or, since the terms can be taken in any order, the sum is $2a+a+3b-5b=3a-2b$, by collecting the like terms.

Ex. 2. Find the sum of $6a^2-6ab+4b^2$, $2b^2-ab-a^2$ and $5ab-9b^2-4a^2$.

The sum is
$$6a^2-6ab+4b^2+2b^2-ab-a^2+5ab-9b^2-4a^2$$
$$=6a^2-a^2-4a^2-6ab-ab+5ab+4b^2+2b^2-9b^2.$$

The terms $6a^2$, $-a^2$ and $-4a^2$ can be combined *mentally*; and we have a^2. Similarly we have $-2ab$, and $-3b^2$.

Thus the required sum is $a^2-2ab-3b^2$.

27. It is best for beginners to place the different sets of 'like' terms in vertical columns; so that the last example would be put down in the following form, the + sign being put before the terms $2b^2$ and $5ab$ which have no signs: and then the sets of 'like' terms can be combined mentally.

$$\begin{array}{r}
6a^2-6ab+4b^2, \\
-\ \ a^2-\ \ ab+2b^2, \\
\underline{-4a^2+5ab-9b^2} \\
a^2-2ab-3b^2
\end{array}$$

EXAMPLES. IV.

Simplify the following by combining 'like' terms:

1. $2a+b+3c+2b+3a+2c+5a+c.$

2. $6a-3b+2c-4a-3c+2b.$

3. $5a^2-3a+6-4a^2+6a+3.$

4. $7a^3-4a+9-3a^2+2a+7-3a^3-16.$

5. $5a^3-4a^2b+3ab^2-5a^3-4a^2b-3b^3.$

6. $3a^2+6ab-4b^2-2a^2-4ab+3b^2-a^2-2ab+b^2.$

Add together

7. $a+b$ and $a-b.$ **8.** $2x-y$ and $2x+y.$

9. $\frac{1}{2}a+\frac{1}{2}b$ and $-\frac{1}{2}a+\frac{1}{2}b.$ **10.** $\frac{1}{3}a+\frac{4}{3}b$ and $\frac{2}{3}b-\frac{1}{3}a.$

11. a^2-a and $a^3+a.$ **12.** $a+a^2+4a^3$ and $2a^3-a^2-4a.$

13. m^2+mn+n^2 and $m^2-mn-n^2.$

14. $3p^2+5pq-6q^2$ and $5q^2-4pq-3p^2$

15. $3a^2-2ab+b^2$ and $a^2-2ab-\frac{3}{2}b^2.$

16. $2a+b-3c,\ 2b+c-3a$ and $2c+a-3b.$

17. $4a-3b-c,\ 4b-3c-a$ and $4c-3a-b.$

18. $4a^2-3ab+b^2,\ 4ab-3b^2+a^2$ and $4b^2-3a^2+ab.$

19. $a-\frac{1}{2}b+\frac{1}{2}c,\ b-\frac{1}{2}c+\frac{1}{2}a$ and $c-\frac{1}{2}a+\frac{1}{2}b.$

20. $4x-2y+1,\ -3x+2-y$ and $x+3y-3.$

21. $-4a-b+2,\ 2+8a-5b$ and $-a-4b-2.$

22. $x^3-2x^2y-2xy^2,\ x^2y-3xy^2-y^3$ and $3xy^2-2y^3-x^3.$

23. $a^3+4b^3-5c^3+3abc,\ b^3+4c^3-5a^3+6abc$ and $c^3+4a^3-5b^3-9abc.$

24. $5a^3-2a^2b+9ab^2+17b^3,\ -2a^3+5a^2b-4ab^2-12b^3,$
$b^3-4ab^2-5a^2b-a^3$ and $2a^2b-2a^3-6b^3-ab^2.$

25. $\frac{2}{3}a^3-\frac{4}{5}a^2b+3b^3,\ a^3-\frac{2}{3}ab^2-\frac{3}{2}b^3$ and $\frac{1}{3}a^2b-\frac{2}{5}ab^2-\frac{1}{6}b^3.$

26. $3x^3-4x+5,\ 2x^2-6x+7,\ 6x^3-2x^2-2x$ and $3+8x-4x^3.$

27. $x^3-3ax^2+5a^2x-a^3,\ \ 2x^3+4ax^2-6a^2x,\ \ 6ax^2-3a^2x+a^3$ and
$-2x^3+4a^2x-5a^3.$

28. $3x^2+y^2-3yz-z^2,\ 2xy-3y^2+3yz$ and $-4x^2-2xy+y^2+z^2.$

29. Shew that, if $x=b+2c-3a$, $y=c+2a-3b$ and $z=a+2b-3c$; then will $x+y+z=0.$

30. Shew that, if $a=5x-3y-2z$, $b=5y-3z-2x$ and $c=5z-3x-2y$; then will $a+b+c=0.$

SUBTRACTION.

28. Since subtraction is the reverse operation to that of addition, to subtract a positive quantity produces a *decrease*, and to subtract a negative quantity produces an *increase*. Hence to subtract a positive quantity we must subtract its absolute value, and to subtract a negative quantity we must add its absolute value.

Thus, to subtract $+4$ from $+10$, we must decrease the amount by 4; we then get $+10-4=+6$. Also, to subtract -4 from $+6$, we must increase the amount by 4; we then get $+6+4=+10$.

Hence $+10-(+4)=+10-4=6$,

and $+6-(-4)=+6+4=10$.

So also $+a-(+b)=+a-b$,

and $+a-(-b)=+a+b$.

We therefore have the following rule for the subtraction of any term :—*to subtract any term affix it to the expression from which it is to be subtracted but with its sign changed.*

29. The rule for subtraction can also be shewn as follows.

Since subtraction is the reverse of addition, an addition and a subtraction of the same quantity produces no effect. Hence a is the same as $a+b-b$. Now if we take $+b$ from $a+b-b$, what is left is $a-b$. So that if we take away $+b$ from a the remainder is $a-b$, that is

$$a-(+b)=a-b.$$

Again, if we take away $-b$ from $a+b-b$, what is left is $a+b$. So that if we take away $-b$ from a the remainder is $a+b$, that is

$$a-(-b)=a+b.$$

Examples. Subtract (i) 3 from -4, (ii) -4 from 3, (iii) -6 from 4, (iv) a from $-b$ and (v) $-a$ from $-b$.

(i) $-4-(+3)=-4-3=-7$; (ii) $3-(-4)=3+4=7$;

(iii) $4-(-6)=4+6=10$; (iv) $-b-(+a)=-b-a$;

(v) $-b-(-a)=-b+a$.

30. Since subtraction and addition are inverse operations, and since we know that to add any algebraical expression as a whole gives the same result as to add its terms separately, it follows that to subtract an algebraical expression as a whole is the same as to subtract the terms in succession.

We have therefore the following

Rule. *To subtract any algebraical expression from any other, write down its terms in succession with their signs changed, after that other.*

Thus, if $2a+b-4c$ be subtracted from $3a-4b+c$, the result is $3a-4b+c-2a-b+4c=3a-2a-4b-b+c+4c=a-5b+5c.$

31. The expression which is to be subtracted is often placed under that from which it is to be taken, 'like' terms being for convenience placed under one another; and the signs of the lower line are changed *mentally* before combining 'like' terms.

Thus the example considered in Art. 30 would be written as follows:

$$\begin{array}{r}3a-4b+c\\ \underline{2a+b-4c}\\ a-5b+5c\end{array}$$

The terms of the result being obtained by combining mentally $3a$ and $-2a$, $-4b$ and $-b$, and c and $+4c$.

As another example, if we have to subtract

$3a^3-4a^2b+2ab^2-b^3$ from $4a^3+a^2b-ab^2$,

the process is written as follows:

$$\begin{array}{l}4a^3+a^2b-ab^2\\ \underline{3a^3-4a^2b+2ab^2-b^3}\\ a^3+5a^2b-3ab^2+b^3\end{array}$$

The terms of the result being obtained by combining $4a^3$ and $-3a^3$, a^2b and $+4a^2b$, $-ab^2$ and $-2ab^2$, 0 and $+b^3$.

32. We have hitherto supposed that the letters used to represent quantities were restricted to positive values; it would, however, be very inconvenient to retain this restriction. In what follows therefore it must always be understood, unless the contrary is expressly stated, that each letter may have any value *positive* or *negative.*

Since any letter may stand for either a positive or a negative quantity, a term preceded by the sign $+$ is not necessarily a positive quantity in reality; such terms are however still to be called *positive terms,* because they are so *in appearance*; and the terms preceded by the sign $-$ are similarly called *negative terms.*

33. We must now carefully examine whether terms can be added and subtracted without knowing whether the letters really represent positive or negative quantities.

Now we have seen in Articles 23 and 28, that when b is really *positive,*

$$+(+b)=+b,\ +(-b)=-b,\ -(+b)=-b \text{ and } -(-b)=+b;$$

and we have to see whether the same laws hold good although b may really be *negative.*

If b be really negative and equal to $-c$, where c is positive; then $+b=+(-c)=-c$, and $-b=-(-c)=+c$, since c is positive. Hence, putting $-c$ for $+b$ and $+c$ for $-b$, the laws expressed above will be true, *although b is negative,* provided

$$+(-c)=-c,\ +(+c)=+c,\ -(-c)=+c \text{ and } -(+c)=-c,$$

are true for all *positive* values of c, and this we know is the case.

Hence *terms are added or subtracted in precisely the same way whether the letters really stand for positive or for negative quantities.*

34. **Def.** The **algebraical difference** between any two quantities a and b is the result obtained by subtracting the *second* from the *first.*

For example, the algebraical difference of 5 and 4 is $5-4=1$, and the algebraical difference of 4 and 5 is $4-5=-1$.

The *algebraical difference* between two quantities depends upon the order in which they are given, and may therefore not be the same as the *arithmetical difference,* which is the result obtained by subtracting the *less* from the *greater.*

The symbol $a \sim b$ is used to denote the *arithmetical difference* of a and b.

Def. One quantity a is said to be **greater than** another quantity b, when the *algebraical difference,* $a-b$, is positive.

From this definition it is easy to see that in the series 1, 2, 3, 4, &c., each number is *greater* than the one before it; and that, in the series -1, -2, -3, -4, &c. each number is *less* than the one before it. Thus 7, 5, 1, 0, -5, -7 are in descending order of magnitude.

EXAMPLES. V.

Subtract

1. $a-b$ from $a+b$. **2.** $2a-3b$ from $3a-2b$.
3. $2x+y$ from $2x-y$. **4.** $\frac{1}{2}x-\frac{1}{3}y$ from $\frac{1}{3}x-\frac{1}{2}y$.
5. $3x-4x^2$ from $4x-3x^2$. **6.** x^2-2 from $1-2x^2$.
7. $4a-2b+3c$ from $2c+4b-3a$.
8. $4a^2-2ab+3b^2$ from $2b^2+4ab-3a^2$.
9. $3x^2-4x+2$ from x^2+6x-7.
10. $3x^3-2x^2+5$ from $3x^3-2x+5$.
11. $a-\frac{1}{2}b-\frac{1}{3}c$ from $b-\frac{1}{2}c-\frac{1}{3}a$.
12. $\frac{1}{4}x^2-\frac{1}{3}xy+\frac{1}{2}y^2$ from $\frac{1}{4}y^2-\frac{1}{3}xy+\frac{1}{2}x^2$.

Find the difference between

13. $a+2b$ and $a-2b$. **14.** $3a-7b$ and $7a-3b$.
15. a^2+ab+b^2 and a^2-ab+b^2. **16.** x^2-3xy and $3x^2-4xy$.
17. $3x^3+5x^2y+4xy^2$ and $4x^2y+6xy^2+7y^3$.

18. $2x^3-7x^2y+9xy^2-4y^3$ and $4x^3-x^2y+9xy^2-4y^3$.

19. What must be added to $2a-3b$ in order that the sum may be $4a-6b$?

20. What must be added to a^2+b^2 in order that the sum may be $2a^2-ab$?

21. What must be added to $5ab-2bc+3ca$ in order that the sum may be $7ab+2ca$?

22. What must be added to $a^2+3b^2+2c^2$ that the sum may be b^2-3a^2?

23. Subtract from $3a-4b$ the sum of $2a+7b$, $-4a-6b$ and $6a-5b$.

24. Subtract from $3x^2-2x+7$ the sum of x^2-x+9, $2x^2+7x-6$ and $3x^2-4x-5$.

25. Subtract the sum of $a^2-4ab-3b^2$, $ab-4b^2-3a^2$ and b^2-4a^2-3ab from $2a^2+2b^2+2c^2$.

BRACKETS.

35. To indicate that any algebraical expression is to be added *as a whole*, it is put between brackets with the + sign prefixed. But, as we have seen, to add any expression we have only to write down the terms in succession with their signs unchanged.

Hence, when a bracket is preceded by a + sign, the bracket may be omitted. Thus

$$+(2a-b+c)=2a-b+c.$$

Conversely any number of terms in an expression may be enclosed in brackets with the sign + placed before the bracket.

For example, we may write $a-2b+c+2d-3e+f$ in the form $a+(-2b+c)+(2d-3e+f)$,

or $$a-2b+(c+2d)+(-3e+f).$$

When the sign of the *first* term in a bracket is + it is generally omitted for shortness, as in the preceding example.

36. To indicate that any algebraical expression is to be subtracted *as a whole*, it is put between brackets and the $-$ sign prefixed. But, as we have seen, to subtract any expression we have only to write down the terms in succession with all their signs changed.

Hence, when a bracket is preceded by the $-$ sign, the brackets may be omitted, provided that the signs of all the terms within the brackets are changed.

Thus $$-(2a-b+c)=-2a+b-c.$$

Conversely any number of terms in an expression may be enclosed in brackets with the sign $-$ placed before the bracket, provided that we change the signs of all the terms which are placed in the brackets.

Thus $a-2b+c+2d-3e+f$ may be written in the form

$$a-(2b-c)-(-2d+3e-f).$$

37. Sometimes brackets are put within brackets: in this case the different pairs of brackets must be of different shapes to prevent confusion.

Thus $a-[b+\{c-(d+e)\}]$; which means that we are to add to b the whole quantity within the brackets marked $\{\ \}$, and then subtract the result from a; and, to find the quantity within the brackets marked $\{\ \}$, we must add d and e, and then subtract the sum from c.

When there are several pairs of brackets they may be removed *one at a time* by the rules of Arts. 35 and 36; and it is best for beginners to remove at every stage the *innermost* bracket.

Thus $$\begin{aligned} a-[b+\{c-(d+e)\}] &= a-[b+\{c-d-e\}] \\ &= a-[b+c-d-e]=a-b-c+d+e. \end{aligned}$$

EXAMPLES. VI.

Simplify the following expressions by removing the brackets and collecting like terms:

1. $(a+b)-(a-b)$. **2.** $a-b-(a+b)$.

3. $a-(b+c)+(b-c-a)$. **4.** $3x-(y-2x)+(z+y-5x)$.

5. $x-\{y-(z-x)\}$. **6.** $a-[a-\{a-(2a-a)\}]$.

7. $1-[2-\{3-(4-5)\}]$. **8.** $a+b-[a-b+\{a+b-(a-b)\}]$.

9. $5-[4+\{5-(4+\overline{5-4})\}]$. **10.** $x-[y-\{z-(x-\overline{y-z})\}]$.

11. $3x-\{2y+5z-\overline{3x+y}\}$.

12. $\{2x-(5y-\overline{3z+7})\}-[4+\{x-(3y+2z+5)\}]$.

13. $[2a-\{3b+(4c-\overline{3b+2a})\}]$. **14.** $x-(y-z)+\{2z-\overline{3y-5x}\}$.

15. $a-2b-\{3a-(b-c)-5c\}$. **16.** $a-[3b+\{3c-(d-b)+a\}-2a]$.

17. $3a-[2b-\{4c-12a-(4b-8c)\}-(6b-12c)]$.

18. $\{2x-(3y-7z)+(3x-2y+9z)\}-\{(y-5z)-(3x-y-2z)+8z\}$.

19. $a^2-(3ab-4b^2)-(2a^2-3ab+6b^2)-\{5b^2-(3ab-\overline{7a^2-b^2})\}$.

20. $(m^2-n^2)-\{3mn-(5n^2-m^2)\}+[n^2-\{3mn-(5m^2-6n^2)\}+8mn]$.

CHAPTER III.

MULTIPLICATION.

38. In Arithmetic, multiplication is first defined to be the taking one number as many times as there are units in another. Thus, to multiply 5 by 4, we take as many fives as there are units in four. As soon, however, as fractional numbers are considered, it is found necessary to modify somewhat the meaning of multiplication, for by the original definition we can only multiply by *whole* numbers. The following is therefore taken as the definition of multiplication:

Def. *To multiply one number by a second we do to the first what is done to unity to obtain the second.*

Thus $\qquad 4$ is $1+1+1+1$;

$\therefore \qquad 5\times 4$ is $5+5+5+5$.

Again, to multiply $\frac{5}{7}$ by $\frac{3}{4}$, we must do to $\frac{5}{7}$ what is done to unity to obtain $\frac{3}{4}$, that is we must divide $\frac{5}{7}$ into four equal parts and take three of those parts. Each of the parts into which $\frac{5}{7}$ is to be divided will be $\dfrac{5}{7\times 4}$, and by taking three of these we get $\dfrac{5\times 3}{7\times 4}$.

So also,

$$\begin{aligned}(-5)\times 4 &= -5+(-5)+(-5)+(-5)\\ &= -5-5-5-5\\ &= -20.\end{aligned}$$

With the above definition multiplication by a negative quantity presents no difficulty.

For example, to multiply 4 by -5. Since to subtract 5 by one subtraction is the same as to subtract five units successively,

$$-5=-1-1-1-1-1;$$

$$\therefore \qquad 4\times(-5)=-4-4-4-4-4$$
$$=-20.$$

Again, to multiply -5 by -4. Since

$$-4=-1-1-1-1;$$

$$\therefore \qquad (-5)\times(-4)=-(-5)-(-5)-(-5)-(-5)$$
$$=+5+5+5+5 \qquad \text{[Art. 28.]}$$
$$=+20.$$

We can proceed in a similar manner for any other numbers whether integral or fractional, positive or negative. Hence we have the following laws:

$$\left.\begin{array}{r} a\times \quad b=+ab\ldots\ldots\text{(i)}\\ (-a)\times \quad b=-ab\ldots\ldots\text{(ii)}\\ a\times(-b)=-ab\ldots\ldots\text{(iii)}\\ (-a)\times(-b)=+ab\ldots\ldots\text{(iv)} \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(\beta).$$

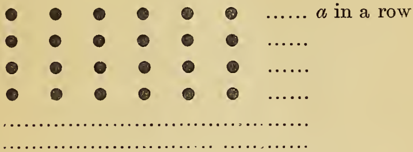
The rule by which we determine the signs of the products is called the **Law of Signs**: this law is sometimes enunciated briefly as follows :—*like signs give* +, *and unlike signs* −.

39. **The factors of a product may be taken in any order.**

It is proved in Arithmetic that when one number, whether integral or fractional, is multiplied by a second, the result is the same as when the second is multiplied by the first.

The proof is as follows:

First, when the numbers are *integers*, a and b suppose, write down a series of rows of dots, putting a dots in each row, and take b rows, writing the dots under one another as in the following arrangement:



b rows.

Then, counting by rows, the whole number of the dots is a repeated b times, that is $a \times b$. Also, counting by columns, the whole number of the dots is b repeated a times, that is $b \times a$. Hence, *when* a *and* b *are integers*,

$$a \times b = b \times a.$$

Next, when the numbers are *fractional*, for example $\frac{5}{7}$ and $\frac{3}{4}$, we prove as in Art. 38 that $\frac{5}{7} \times \frac{3}{4} = \frac{5 \times 3}{7 \times 4}$. And, by the above proof for integers, $\frac{5 \times 3}{7 \times 4} = \frac{3 \times 5}{4 \times 7}$; hence

$$\tfrac{5}{7} \times \tfrac{3}{4} = \tfrac{3}{4} \times \tfrac{5}{7}.$$

Hence $ab = ba$ for all *positive* values of a and b; and being true for any positive values of a and b, it must be true for all values, whether positive or negative; for, from the preceding Article, the absolute value of the product is independent of the signs, and the sign of the product is independent of the order of the factors.

Hence for all values of a and b we have

$$ab = ba \text{ .... (i).}$$

If in the above arrangement of dots we put c in the place of each of the dots, the whole number of c's will be ab; also the number of c's in the first row will be a, and this is repeated b times. Hence, when a and b are integers, c repeated ab times gives the same result as c repeated a times and this repeated b times. So that to multiply by any two *whole numbers* in succession gives the same result as to multiply at once by their product; and the proposition can, as before, be then proved to be true without restriction to whole numbers or to positive values.

Thus, for all values of a, b and c, we have

$$a \times b \times c = a \times (bc) \text{..................(ii).}$$

From (i) and (ii) it follows that *the factors of a product may be taken in any order without altering the result*, however many factors there may be.

40. Since the factors of a product may be taken in any order, we are able to simplify many products.

For example:

$$3a \times 4a = 3 \times 4 \times a \times a = 12a^2,$$

$$(-3a) \times (-4b) = +3a \times 4b = +3 \times 4 \times a \times b = 12ab,$$

$$(ab)^2 = ab \times ab = a \times a \times b \times b = a^2b^2,$$

$$(\surd 2a)^2 = \surd 2a \times \surd 2a = \surd 2 \times \surd 2 \times a \times a = 2a^2.$$

Although the order of the factors in a product is indifferent, a factor expressed in figures is always put first, and the letters are usually arranged in alphabetical order.

41. By definition, $a^2 = aa$, $a^3 = aaa$, $a^4 = aaaa$, &c.

Hence $a^2 \times a^3 = aa \times aaa = a^5 = a^{2+3}$;

also $a^3 \times a^5 = aaa \times aaaaa = a^8 = a^{3+5}$;

and $a \times a^4 = a \times aaaa = a^5 = a^{1+4}$.

In the above examples we see that *the index of the* *product of two powers of the same letter is equal to the sum of the indices of the factors.* We can prove in the following manner that the above is true whenever the indices are positive integers:

Since, by definition,

$$a^m = aaaa \dots \text{ to } m \text{ factors,}$$

and $$a^n = aaaa \dots \text{ to } n \text{ factors;}$$

$$\begin{aligned} \therefore\ a^m \times a^n &= (aaa \dots \text{ to } m \text{ factors}) \times (aaa \dots \text{to } n \text{ factors}) \\ &= aaa \dots \text{ to } m+n \text{ factors} \\ &= a^{m+n}, \qquad \text{by definition.} \end{aligned}$$

Thus, for any positive values of m and n

$$a^m \times a^n = a^{m+n}.$$

This result is called the **Index Law.**

42. **Product of monomial expressions.** The results arrived at in the preceding articles will enable us to find the product of any monomial expressions. These results are:—

(i) The sign of the product of two quantities is + when the factors are both positive or both negative; and the sign of the product is – when one factor is positive and the other negative.

(ii) The factors of a product may be taken in any order.

(iii) The index of the product of any two powers of the same quantity is the sum of the indices of the factors.

Ex. 1. Multiply $3a^2b^2$ by $6a^2b^3$.

$$\begin{aligned} 3a^2b^2 \times 6a^2b^3 &= 3 \times 6 \times a^2 \times a^2 \times b^2 \times b^3, && \text{from (ii),} \\ &= 18a^{2+2}b^{2+3} = 18a^4b^5, && \text{from (iii).} \end{aligned}$$

Ex. 2. Multiply $-3a^2b$ by $-5ab^5$.

$$\begin{aligned} (-3a^2b) \times (-5ab^5) &= +3a^2b \times 5ab^5, && \text{from (i),} \\ &= 3 \times 5 \times a^2 \times a \times b \times b^5, && \text{from (ii).} \\ &= 15a^{2+1}b^{1+5} = 15a^3b^6, && \text{from (iii).} \end{aligned}$$

Ex. 3. Multiply $2a^2b^3c^4$ by $-5a^5b$.

The whole work of finding the product of two monomial expressions can be performed *mentally* and the result written down at once. First put down the sign of the result; then multiply the numerical coefficients to find the numerical coefficient of the product; then take each letter which occurs to a power whose index is the sum of the indices of that letter in the factors. Thus the product of

$$2a^2b^3c^4 \text{ and } -5a^5b \text{ is } -10a^7b^4c^4.$$

Ex. 4. Find the cube of $2a^2b$.

The cube is $2a^2b \times 2a^2b \times 2a^2b = 8a^6b^3$.

EXAMPLES. VII.

Multiply

1. $3a$ by $6a$. **2.** $5a^2$ by $7a$. **3.** $2a^3$ by $5a^2$.
4. ab by a^2b^3. **5.** $3a^2b$ by $2ab^2$. **6.** $4ab^3$ by $7a^4b^2$.
7. $3a^2bc^3$ by $6ab$. **8.** $5ab^3c^2$ by $3ab^2$. **9.** $2a^3b^2c$ by abc.
10. $2a$ by $-4b$. **11.** $3b$ by $-4a$. **12.** a^2 by $-a$.
13. $-6a^3b$ by $4ab$. **14.** $-2ab^3$ by $-7a^6b^2$. **15.** $-3a^2bc^2$ by $6ab^2c^5$.
16. $-3ab^2c$ by $2ab^3c^2$. **17.** $-2xy^4$ by $-5x^4y^3$.
18. $2ax^2y^3z$ by $-5a^2xy^4z^3$. **19.** $6a^3b^2c^3x^5y^4z$ by $-12ab^3c^4x^2y^3z^5$.

20. Find the values of $(-a)^2$, $(-a)^3$ and $(-a)^4$.

21. Find the values of $(-x^2)^2$, $(-x^2)^3$ and $(-x^2)^4$.

22. Find the values of $(-ab)^2$, $(-ab)^3$ and $(-ab)^4$.

23. Find the values of $(a^2b^3)^2$, $(a^2b^3)^3$ and $(a^2b^3)^4$.

24. Find the values of $(-2a^3b^4)^2$, $(-2a^3b^4)^3$ and $(-2a^3b^4)^4$.

25. Find the squares of $3ab^2c^3$, $-2a^3bc^4$ and $-4a^2b^3c^5$.

26. Find the cubes of a^2, $-a^4$, ab and $-a^2b$.

27. Find the cubes of $2ab^2$, $-3a^3b^2$, $-4ab^5$ and $-7a^2b^5c^4$.

28. Find the values of $(-a)^2 \times (-b)^3$, $(-2a)^3 \times (a^2)^2$, $(-ab^2)^3 \times (-a^2b)^3$ and $(a^2b^2)^3 \times (-ab^3)^4$.

If $a=2$, $b=-3$, $c=-1$, $d=0$, find the numerical values of

29. $6ab$. **30.** $4abc$. **31.** $8a^2b^2c^2$.
32. $ab+bc+ca$. **33.** $a^3+b^3+c^3+d^3$. **34.** $a^2b^2+b^2c^2+c^2d^2$.
35. $3a^2b^2c^3+5b^2c^4$ **36.** $2ab^2+3bc^2+4cd^2$. **37.** $(a+b)(c+d)$.
38. $(a-b)^2(c-d)^2$. **39.** $(a^2+bc)(b^2+cd)$. **40.** $(a-b)^3(c-a)^3$.

43. We now proceed to the multiplication of multinomial expressions.

We first observe that any multinomial expression can be put in the form

$$a + b + c + \&c.,$$

where a, b, c, &c. may be *any quantities positive or negative.*

For example, the expression $3x^2y - \frac{5}{2}xy^2 - 7xyz$, which by Art. 23 is the same as $3x^2y + (-\frac{5}{2}xy^2) + (-7xyz)$, takes the required form if we put a for $3x^2y$, b for $-\frac{5}{2}xy^2$, and c for $-7xyz$.

It therefore follows that in order to prove any theorem to be true for *any algebraical expression*, it is only necessary to prove it for the expression $a + b + c +$ &c. where a, b, c, &c. are supposed to have any *positive or negative values.*

44. **Product of a multinomial expression and a monomial.** Consider the product $(a + b)c$, where a, b and c have any values whatever.

If c be a *positive integer*, and a and b have any values whatever; then

$$\begin{aligned}(a + b)c &= (a + b) + (a + b) + (a + b) + \dots \text{ repeated } c \text{ times,}\\ &= a + b + a + b + a + b + \dots\\ &= a + a + a + \dots \text{ repeated } c \text{ times,}\\ &\quad + b + b + b + \dots \text{ repeated } c \text{ times,}\\ &= ac + bc.\end{aligned}$$

Hence *when c is a positive integer*, we have

$$(a + b)c = ac + bc.$$

Since division is the inverse of multiplication, it follows that when d is a positive integer

$$(a + b) \div d = a \div d + b \div d.$$

And since the operations of multiplication and division to be performed on $a+b$ can be performed on a and b *separately*, this must also be the case for the complex operation denoted by a fraction $\frac{m}{n}$.

Thus $(a+b)c = ac + bc$ for all *positive values* of c; and being true for any positive value of c, it must also be true for any negative value. For, if

$$(a+b)\,c = ac + bc,$$

then
$$-(a+b)\,c = -ac - bc;$$

and therefore
$$(a+b)(-c) = a(-c) + b(-c).$$

Hence, for *all values* of a, b and c, we have

$$(a+b)\,c = ac + bc.$$

45. Since $(a+b)c = ac + bc$ for *all values* of a, b and c, it will be true when $x+y$ is put in place of a.

Hence $\{(x+y)+b\}\,c = (x+y)\,c + bc = xc + yc + bc$;

$$\therefore \quad (x+y+b)\,c = xc + yc + bc.$$

And similarly

$$(x+y+z+p+\ldots)\,c = xc + yc + zc + pc + \ldots,$$

however many terms there may be in the expression

$$x+y+z+p+\ldots$$

Thus *the product of any multinomial expression by a monomial is the sum obtained by multiplying the separate terms of the multinomial expression by the monomial.*

Ex. 1. Multiply $a^2 + a^3$ by a.

The result is $a^2 \times a + a^3 \times a = a^3 + a^4$.

Ex. 2. Multiply $a - b$ by c.

The result is $a \times c + (-b) \times c = ac - bc$.

Ex. 3. Multiply $-3x^2$ by $x-1$.

$$-3x^2 \times (x-1) = (x-1) \times (-3x^2) = -3x^3 + 3x^2.$$

EXAMPLES. VIII.

Multiply

1. $a+b$ by 3.
2. $2a-b$ by 4.
3. $3a-4b$ by 6.
4. a^2+a by a.
5. a^2-a by a^3.
6. a^2+1 by $3a^3$.
7. $4a^2-5a+1$ by a^4.
8. $2a^2-3a-4$ by $-3a^3$.
9. $2a^2-3ab+2b^2$ by a^2b^2.
10. $bc+ca+ab$ by abc.
11. $2x^3-3x^2+5x-4$ by $-5x^2$.
12. $4-3x^2+3x^3-4x^4$ by $-6x^3$.
13. $-5ab$ by $3a^2-2ab+7b^2$.
14. $-6a^3b^4$ by $2a^3-3a^2b-5b^2$.

Simplify

15. $2(a-b)+4(a+b)$.
16. $\frac{1}{2}(b-2c)-\frac{3}{4}(c-2b)$.
17. $c(a+b)-c(a-b)$.
18. $7a(b-c)-2b(a-c)$.
✓**19.** $a^2b^2(c^2-d^2)+c^2d^2(a^2-b^2)+b^2c^2(d^2-a^2)$.
20. $2\{3ab-4a(c-2b)\}$.
21. $3a-2[b-\{2c-6a-2(b-2c)\}-3(b-2c)]$.
22. $4a^3-[(2b^3-3c^3)-6bc(b+c)+3c(c^2+2b^2)]$.
23. $7ac-2\{2c(a-3b)-3(5c-2b)a\}$.

46. **Product of two multinomial expressions.** We have now to consider the most general case of multiplication, namely the multiplication of any two multinomial expressions.

We have to find

m.

$$(a+b+c+\ldots)\times(x+y+z\ldots);$$

and, from Art. 43, this includes all possible cases.

Put M for $x+y+z+\ldots$; then, by the last Article, we have

$$(a+b+c+\ldots)M=aM+bM+cM+\ldots$$
$$=Ma+Mb+Mc+\ldots\ [\text{Art. } 39].$$
$$=(x+y+z+\ldots)a+(x+y+z+\ldots)b+(x+y+z+\ldots)c+\ldots$$
$$=ax+ay+az+\ldots+bx+by+bz+\ldots+cx+cy+cz+\ldots+\ldots.$$

Hence $(a+b+c+\ldots)(x+y+z+\ldots)$

$$=ax+ay+az+\ldots+bx+by+bz+\ldots+cx+cy+cz+\ldots.$$

Thus *the product of any two multinomial expressions is the sum of the products obtained by multiplying every term of the multiplicand by every term of the multiplier.*

For example,

$$(a + b)(c + d) = ac + bc + ad + bd.$$

Also

$$(2a + 5b)(3a + 2b) = 2a \times 3a + 5b \times 3a + 2a \times 2b + 5b \times 2b$$
$$= 6a^2 + 15ab + 4ab + 10b^2 = 6a^2 + 19ab + 10b^2.$$

Again, to find $(a - b) \times (c - d)$; we must first write this in the form $\{a + (-b)\}\{c + (-d)\}$, and we then have for the product

$$ac + (-b)c + a(-d) + (-b)(-d),$$

which by Art. 38 is equal to

$$ac - bc - ad + bd.$$

Note. In the rule given above for the multiplication of two algebraical expressions it must be borne in mind that the *terms include the prefixed signs.*

47. The following are important examples:—

(i) $(a + b)^2 = (a + b)(a + b) = aa + ba + ab + bb$;

$$\therefore \quad (a + b)^2 = a^2 + 2ab + b^2.$$

Hence, *the square of the sum of any two quantities is equal to the sum of their squares plus twice their product.*

(ii) $(a - b)^2 = (a - b)(a - b) = aa + (-b)a + a(-b)$
$+ (-b)(-b) = a^2 - ab - ab + b^2$;

$$\therefore \quad (a - b)^2 = a^2 - 2ab + b^2.$$

Hence, *the square of the difference of any two quantities is equal to the sum of their squares minus twice their product.*

(iii) $(a + b)(a - b) = aa + ba + a(-b) + b(-b)$
$= a^2 + ab - ab - b^2$;

$$\therefore \quad (a + b)(a - b) = a^2 - b^2.$$

Hence, *the product of the sum and difference of any two quantities is equal to the difference of their squares.*

48. It is usual to exhibit the process of multiplication in the following convenient form.

$$\begin{array}{lllll}
a^2 & +2ab & -b^2 & & \\
a^2 & -2ab & +b^2 & & \\
\hline
a^4 & +2a^3b & -a^2b^2 & & \\
 & -2a^3b & -4a^2b^2 & +2ab^3 & \\
 & & a^2b^2 & +2ab^3 & -b^4 \\
\hline
a^4 & & -4a^2b^2 & +4ab^3 & -b^4
\end{array}$$

The multiplier is placed under the multiplicand and a line is drawn. The successive terms of the multiplicand, namely a^2, $+2ab$ and $-b^2$, are multiplied by a^2, the first term on the left of the multiplier; and the products a^4, $+2a^3b$ and $-a^2b^2$ which are thus obtained are put in a horizontal row. The terms of the multiplicand are then multiplied by $-2ab$, the second term of the multiplier, and the products thus obtained are put in another horizontal row, the terms being so placed that 'like' terms are under one another. The terms of the multiplicand are then multiplied by b^2, the last term of the multiplier, and the products thus obtained are put in a third horizontal row, 'like' terms being again placed under one another. The final result is then obtained by adding the rows of partial products; and this final sum can be readily written down, since the different sets of 'like' terms are in vertical columns.

49. The following are additional examples of multiplications arranged as in the preceding article.

$$\begin{array}{lll}
a & +b & \\
a & +b & \\
\hline
a^2 & +ab & \\
 & +ab & +b^2 \\
\hline
a^2 & +2ab & +b^2
\end{array}
\qquad\qquad
\begin{array}{lll}
a & -b & \\
a & -b & \\
\hline
a^2 & -ab & \\
 & -ab & +b^2 \\
\hline
a^2 & -2ab & +b^2
\end{array}
\qquad\qquad
\begin{array}{lll}
a & +b & \\
a & -b & \\
\hline
a^2 & +ab & \\
 & -ab & -b^2 \\
\hline
a^2 & & -b^2
\end{array}$$

$$\begin{array}{llll}
a+\sqrt{2b} & & & \\
a-\sqrt{2b} & & & \\
\hline
a^2+\sqrt{2}ba & & & \\
\quad -\sqrt{2}ba & -2b^2 & & \\
\hline
a^2 & -2b^2 & &
\end{array}
\qquad\qquad
\begin{array}{llllll}
a & +b & +c & & & \\
a & +b & +c & & & \\
\hline
a^2 & +ab & +ac & & & \\
 & +ab & & +b^2 & +bc & \\
 & & +ac & & +bc & +c^2 \\
\hline
a^2 & +2ab & +2ac & +b^2 & +2bc & +c^2
\end{array}$$

EXAMPLES. IX.

Multiply

1. $x+2y$ by $x-2y$.
2. $a-3b$ by $a+3b$.
3. $2x+3y$ by $3x-2y$.
4. $5a+4b$ by $a-b$.
5. $x+7$ by $x+6$.
6. $x-7$ by $x-6$.
7. $x+7$ by $x-6$.
8. $a+9$ by $a-5$.
9. $2x-4$ by $2x+6$.
10. $3x-7$ by $2x-1$.
11. $2y+5b$ by $3y-4b$.
12. $3m^2-1$ by $3m^2+1$.
13. $2m^2+5n^2$ by $2m^2-5n^2$.
14. $a+\frac{1}{2}b$ by $a-\frac{1}{2}b$.
15. $2a+\frac{1}{3}b$ by $3a+\frac{1}{2}b$.
16. $\frac{1}{4}a-\frac{1}{3}b$ by $\frac{1}{3}a-\frac{1}{4}b$.
17. x^2+x+1 by $x-1$.
18. x^2-x+1 by $x+1$.
19. a^2+ab+b^2 by $a-b$.
20. a^2-ab+b^2 by $a+b$.
21. $4a^2+6ab+9b^2$ by $2a-3b$.
22. $16p^2+20pq+25q^2$ by $4p-5q$.
23. $x^3-3ax^2+2a^2x$ by $x+3a$.
24. $a^3-4a^2b+6ab^2$ by a^2+4ab.
25. x^3-3x^2+2x+1 by x^2+3x+2.
26. x^3+x^2-2x+1 by x^2-x+2.
27. x^2+xy+y^2 by x^2-xy+y^2.
28. $a^4+a^2b^2+b^4$ by $a^4-a^2b^2+b^4$.
29. $2x^3-3x^2y+2xy^2+y^3$ by $x^2+3xy+2y^2$.
30. $x^3-4x^2y+6xy^2-3y^3$ by $3x^2-4xy+5y^2$.

50. If in any expression consisting of several terms which contain different powers of the same letter, the term which contains the highest power of that letter be put first on the left, the term which contains the next highest power be put next, and so on; the terms, if any, which do not contain the letter being put last; then the whole expression is said to be *arranged according to descending powers* of that

letter. Thus, the expression $a^3 + a^2b + ab^2 + b^3$, is arranged according to descending powers of the letter a. In like manner we say that the above expression is *arranged according to ascending powers* of the letter b.

51. Although it is not *necessary* to arrange the terms either of the multiplicand or of the multiplier in any particular order, it will be found convenient to arrange *both* expressions either according to descending or according to ascending powers of the same letter: some trouble in the arrangement of the different sets of like terms of the product in vertical columns will thus be avoided. Hence before beginning to find the product of two expressions, *it is often desirable to rearrange the terms.*

52. A term which is the product of n letters is said to be of **n dimensions**, or of the **nth degree.** Numerical factors are not to be counted in reckoning the number of dimensions. Thus abc is of three dimensions, or of the third degree; and $5a^2b^2c$, that is $5aabbc$, is of five dimensions, or of the fifth degree.

53. The **degree of an expression** is the degree of that term which is of highest dimensions.

In estimating the degree of a term, or of an expression, we sometimes take into account only a particular letter, or particular letters: thus we say that $ax^2 + bx + c$ is of the *second degree in* x, or is a **quadratic expression** in x; also that $ax^2y + bxy^2$ is of the second degree *in* x, and of the third degree *in* x *and* y.

When all the terms of an expression are of the same dimensions, the expression is said to be **homogeneous.** Thus $a^3 + 3a^2b - 5b^3$ is a homogeneous expression, every term being of three dimensions.

EXAMPLES. X.

Arrange the following expressions according to descending powers of a:

1. $a^2+b^2-2ab.$ **2.** $2-4a^2+5a-6a^3.$

3. $a^3+b^3+a^2b+ab^2.$ **4.** $5a^2-4-6a^3-2a.$

5. $a^3+b^3+c^3-3abc.$

6. $a^3+b^3+c^3+a^2b+ab^2+a^2c+ac^2+b^2c+bc^2.$

7. What are the degrees of the above expressions, and which are homogenous expressions?

Bracket together the different powers of x in each of the following expressions:

8. $x^2+ax+bx+ab.$ **9.** $x^2-ax-bx-ab.$

10. $x^3+ax^2+bx^2+cx^2+bcx+cax+abx+abc.$

11. $ax^2+bx^2+cx^2+bcx+cax+abx+abc.$

12. $x^2(y-z)+y^2(z-x)+z^2(x-y).$

13. Simplify $(b+c-a)x+(c+a-b)x+(a+b-c)x.$

14. Simplify $(b-c)x+(c-a)x+(a-b)x.$

15. Simplify $\{(b-a)x+(c-d)y\}+\{(a+b)x+(c+d)y\}.$

54. **Product of homogeneous expressions.** The product of any two homogeneous expressions must be homogeneous; for the different terms of the product are obtained by multiplying any term of the multiplicand by any term of the multiplier, and the number of dimensions in the product of any two monomial quantities is clearly the sum of the number of dimensions in the separate quantities; hence if all the terms of the multiplicand are of the same degree, as also all the terms of the multiplier, it follows that all the terms of the product are of the same degree.

The fact that two expressions which are to be multiplied are homogeneous should in all cases be noticed by a student; and if the product obtained is not homogeneous, it is at once seen that there is an error.

55. We now return to the three important cases of multiplication considered in Art. 47, namely,

$$(a+b)^2 = a^2 + 2ab + b^2 \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$
$$(a-b)^2 = a^2 - 2ab + b^2 \ldots\ldots\ldots\ldots\ldots \text{(ii)},$$
$$(a+b)(a-b) = a^2 - b^2 \ldots\ldots\ldots\ldots\ldots \text{(iii)}.$$

Def. A general result expressed by means of symbols is called a **formula.**

Since the laws from which the above formulae were deduced were proved to be true for all algebraical quantities whatever, we may substitute for a and for b any other algebraical quantities, or algebraical expressions, and the results will still hold good.

We give some examples of results obtained by substitution in the above formulae.

In the first place, let us put $-b$ in the place of b in (i); we then have

$$\{a+(-b)\}^2 = a^2 + 2a(-b) + (-b)^2,$$

that is, $$(a-b)^2 = a^2 - 2ab + b^2.$$

Thus (ii) is seen to be really included in (i).

Now put $b+c$ in the place of b in (i); we then have

$$\{a+(b+c)\}^2 = a^2 + 2a(b+c) + (b+c)^2;$$
$$\therefore \{a+b+c\}^2 = a^2 + 2ab + 2ac + b^2 + 2bc + c^2.$$

Thus $\{a+b+c\}^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc \ldots$(iv).

Now put $-c$ for c in (iv), and we have

$$\{a+b+(-c)\}^2 = a^2 + b^2 + (-c)^2 + 2ab + 2a(-c) + 2b(-c);$$
$$\therefore (a+b-c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc.$$

Again put $b+c$ in the place of b in (iii); we then have

$$\{a+(b+c)\}\{a-(b+c)\} = a^2 - (b+c)^2.$$

The following are additional examples of products which can be written down at once.

$$(a^2+b^2)(a^2-b^2)=(a^2)^2-(b^2)^2=a^4-b^4,$$

$$(a-b+c)(a+b-c)=\{a-(b-c)\}\{a+(b-c)\}$$

$$=a^2-(b-c)^2=a^2-(b^2-2bc+c^2)=a^2-b^2+2bc-c^2,$$

$$(a^2+ab+b^2)(a^2-ab+b^2)=(\overline{a^2+b^2}+ab)(\overline{a^2+b^2}-ab)$$

$$=(a^2+b^2)^2-a^2b^2=a^4+2a^2b^2+b^4-a^2b^2=a^4+a^2b^2+b^4.$$

56. **Square of any multinomial expression.** We have found in the preceding Article, and also in Art. 49, the square of the sum of *three* algebraical quantities; and we can by the same methods obtain the square of the sum of more than three quantities. The square of the sum of any number of algebraical quantities may also be found in the following manner.

Suppose we wish to find

$$(a+b+c+d+\ldots)(a+b+c+d+\ldots).$$

We know that the product of any two algebraical expressions is equal to the sum of the partial products obtained by multiplying every term of the multiplicand by every term of the multiplier. If we multiply the term a of the multiplicand by the term a of the multiplier, we obtain the term a^2 of the product: we similarly obtain the terms b^2, c^2, &c. We can multiply any term, say b, of the multiplicand by any different term, say d, of the multiplier; and we thus obtain the term bd of the product. But we also obtain the term bd of the product by multiplying the term d of the multiplicand by the term b of the multiplier, and we can obtain the term bd in no other way, so that every such term as bd, in which the letters are different, occurs *twice* in the product. The required product is therefore the sum of the squares of all the quantities a, b, c, d &c., together with twice the product of every pair.

Thus, *the square of the sum of any number of algebraical quantities is equal to the sum of their squares together with twice the product of every pair.*

Ex. 1. Find the square of $a+b+c$. The squares of the separate terms are a^2, b^2, c^2. The products of the different pairs of terms are ab, ac and bc.

Hence $(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$.

Ex. 2. Find the square of $a+2b-3c$.

The required square

$$= a^2+(2b)^2+(-3c)^2+2a\,(2b)+2a\,(-3c)+2\,(2b)\,(-3c)$$
$$= a^2+4b^2+9c^2+4ab-6ac-12bc.$$

Ex. 3. Find $(a-b+c-d)^2$.

$$(a-b+c-d)^2 = a^2+(-b)^2+c^2+(-d)^2+2a\,(-b)+2ac$$
$$+2a\,(-d)+2\,(-b)\,c+2\,(-b)\,(-d)+2c\,(-d)$$
$$= a^2+b^2+c^2+d^2-2ab+2ac-2ad-2bc+2bd-2cd.$$

After some practice the intermediate steps can be omitted and the final result written down at once. To ensure taking twice the product of *every* pair it is best to take twice the product of each term and of every term which *follows* it.

EXAMPLES. XI.

Write down the squares of the following expressions:—

1. $2a+b$.
2. $4a+3b$.
3. $3a-b$.
4. $5a-6b$.
5. a^2-5ab.
6. $2a^2-3ab$.
7. $-3xy+2y^2$.
8. $4x^2-7y^2$.
9. $a-b-c$.
10. $2a+2b-c$.
11. $4a+2b-3c$.
12. $2a-5b-3c$.
13. x^2+x+1.
14. x^2-x+1.
15. x^2-xy+y^2.
16. $x^4+x^2y^2+y^4$.
17. $a-b-c+d$.
18. $2a-2b-3c+3d$.
19. $3a+2b-4c+d$.
20. $5a+b-4c-3d$.
21. x^3+x^2+x+1.
22. x^3-x^2+x-1.
23. $2x^3-x^2y+xy^2-3y^3$.
24. $2x^3-x^2y+2xy^2-y^3$.

Multiply

25. $x-y+z$ by $x-y-z$.
26. $x-2y+4z$ by $x-2y-4z$.
27. $3x-y-5z$ by $3x+y-5z$.
28. $-x+2y-3z$ by $x+2y-3z$.
29. x^2+xy+y^2 by x^2-xy+y^2.
30. $3x^2-xy+2y^2$ by $3x^2+xy+2y^2$.
31. x^2-x+7 by x^2-x-7.
32. $2x^2-3x+7$ by $2x^2+3x+7$.
33. $a+b+c+d$ by $a+b-c-d$.
34. $2a-3b+2c-4d$ by $2a-3b-2c+4d$.
35. $2a+3b+c-2d$ by $2a-3b-c-2d$.
36. $a-3b-4c+d$ by $a+3b-4c-d$.

57. **Continued Products.** The continued product of several algebraical expressions is obtained by finding the product of any two of the expressions, and then multiplying this product by the third expression.

Ex. 1. Find the continued product $(x+a)(x+b)(x+c)$.

The process is written as under:—

$$\begin{array}{l}
x+a \\
x+b \\ \hline
x^2+ax \\
\qquad bx+ab \\ \hline
x^2+(a+b)\,x+ab \\
x+c \\ \hline
x^2+(a+b)\,x^2+abx \\
\qquad\qquad\qquad cx^2+c\,(a+b)\,x+abc \\ \hline
x^3+(a+b+c)\,x^2+(ab+ac+bc)\,x+abc
\end{array}$$

If $a=b=c$, we have $(x+a)^3=x^3+3ax^2+3a^2x+a^3$.

The above result is arranged in a way which is frequently required, namely according to powers of x, and all the terms which contain the same powers of x are collected together.

Ex. 2. Find the continued product of $x^2+a^2, x+a$ and $x-a$.

The factors can be taken in any order; hence the required product

$$=(x-a)(x+a)(x^2+a^2)=(x^2-a^2)(x^2+a^2)=x^4-a^4.$$

Ex. 3. Find $(x-a)^2(x+a)^2$.

Since factors can be taken in any order,

$$(x-a)^2(x+a)^2=(x-a)(x+a)(x-a)(x+a)=\{(x-a)(x+a)\}^2$$
$$=(x^2-a^2)^2=x^4-2a^2x^2+a^4.$$

58. **Powers of a Binomial.** We shall in a succeeding chapter shew how to write down any power of a binomial expression, by a formula which is called the **Binomial Theorem.** The square and the cube of a binomial should however be learnt at this stage. These are given by

$$(a\pm b)^2=a^2\pm 2ab+b^2,$$

and

$$(a\pm b)^3=a^3\pm 3a^2b+3ab^2\pm b^3.$$

EXAMPLES. XII.

Multiply

1. $3a^2+ab-b^2$ by $a^2-2ab-3b^2$. **2.** $x^2-xy-3y^2$ by $3x^2+xy-y^2$.

3. $5x^3-4x^2y+3xy^2$ by $x^2+4xy+6y^2$.

4. $x^3-7x^2y+3xy^2$ by $x^4+7x^3y-3x^2y^2$.

5. $3x^3-7x^2+5x-3$ by $2x^3+7x^2-5x+4$.

6. a^3+3a^2-7a+6 by $3a^3-a^2+5a-4$.

7. $\frac{3}{2}x^2-\frac{5}{4}xy+y^2$ by $\frac{1}{2}x^2-\frac{1}{4}xy-y^2$.

8. $\frac{1}{4}x^2-\frac{2}{3}xy+12y^2$ by $12x^2+\frac{2}{3}xy-\frac{1}{4}y^2$.

9. $a^2+b^2+c^2-bc-ca-ab$ by $a+b+c$.

10. $4a^2+9b^2+c^2-3bc-2ca-6ab$ by $2a+3b+c$.

11. $a^2+b^2+c^2+bc+ca-ab$ by $a+b-c$.

12. $9a^2+b^2+9c^2+3bc-9ca+3ab$ by $3a-b+3c$.

13. x^2+xy+y^2 by y^2-xy+x^2. **14.** $a^2-2ab+4b^2$ by $4b^2+2ab+a^2$.

15. $ax+a^2x^2+a^3x^3$ by a^2x^2-ax+1.

16. x^2+1, $x+1$ and $x-1$. **17.** a^2+x^2, $a+x$, and $a-x$.

18. x^2+4y^2, $x+2y$ and $x-2y$. **19.** $9x^2+25y^2$, $3x+5y$ and $3x-5y$.

20. x^4+y^4, x^2+y^2, $x+y$ and $x-y$.

21. $(a^2+b^2)^2$, $(a+b)^2$ and $(a-b)^2$.

22. $(x^2+x+1)^2$ and $(x^2-x+1)^2$.

23. x^2-xy+y^2, x^2+xy+y^2 and $x^4-x^2y^2+y^4$.

24. $a^4-a^2b^2+b^4$, a^2+ab+b^2 and a^2-ab+b^2.

25. x^2-ax+a^2, x^2+ax+a^2, $a+x$ and $a-x$.

26. $x+y+z$, $-x+y+z$, $x-y+z$ and $x+y-z$.

Find the following cubes

27. $(2a+3b)^3$. **28.** $(2a-3b)^3$. **29.** $(3a-2b)^3$.

30. $(a+b+c)^3$. **31.** $(a-b+c)^3$. **32.** $(a-b-c)^3$.

33. Shew that

(i) $(2x+1)^2+(x-1)^2=4x^2+(x+1)^2+1$.

(ii) $(2x+1)^2+(x+2)^2=(x-2)^2+4x(x+3)+1$.

(iii) $(x^2+x+1)^2+(x^2-x+1)^2=2(x^4+3x^2+1)$.

34. Shew that

$$a^3-b^3=(a-b)(a^2+ab+b^2)=(a-b)^3+3ab(a-b)$$
$$=(a+b)^3-3ab(a+b)-2b^3.$$

35. Shew that

$(x+y)(x+z)+(y+z)(y+x)+(z+x)(z+y)-(x+y+z)^2=yz+zx+xy.$

36. Shew that $(b-c)^3+(c-a)^3+(a-b)^3-3(b-c)(c-a)(a-b)=0.$

37. Simplify $(x+y+z)^2-(-x+y+z)^2+(x-y+z)^2-(x+y-z)^2.$

38. Simplify

$(x+y+z)(x^2+y^2+z^2)-yz(y+z)-zx(z+x)-xy(x+y).$

39. Shew that $(b+c)(c+a)(a+b)+abc=(a+b+c)(bc+ca+ab).$

40. Shew that, if $x=2y+5z$; then will $x^3=8y^3+125z^3+30xyz.$

41. Shew that, if $x=b+c-2a$, $y=c+a-2b$, and $z=a+b-2c$; then

$x^2+y^2+z^2+2yz+2zx+2xy=0.$

42. Shew that $a^3+b^3+c^3+3(b+c)(c+a)(a+b)=(a+b+c)^3.$

43. Shew that $a^2(b+c)^2+b^2(c+a)^2+c^2(a+b)^2+2abc(a+b+c)$

$=2(bc+ca+ab)^2.$

44. Shew that $(a+b)^3+3c(a+b)^2+3c^2(a+b)+c^3$

$=(b+c)^3+3a(b+c)^2+3a^2(b+c)+a^3.$

45. Shew that

$\{(a+b)x^2-(a^2+b^2)x+a^3+b^3\}\{(a-b)x^2-(a^2-b^2)x+a^3-b^3\}$

$=(a^2-b^2)x^4-2(a^3-b^3)x^3+3(a^4-b^4)x^2-2(a^5-b^5)x+a^6-b^6.$

CHAPTER IV.

DIVISION.

59. In multiplication we have two factors given, and we have to find their product. In division we have the product and one factor given, and we have to determine the other factor, so that to divide a by b is to find a quantity c such that $b \times c = a$.

60. Since division is the inverse operation to that of multiplication, and successive multiplications can be performed in any order [Art. 39], it follows that successive divisions can be performed in any order.

Thus $$a \div b \div c = a \div c \div b.$$

It also follows from Art. 39, that to divide by any quantities in succession gives the same result as to divide at once by their product.

Thus $$a \div b \div c = a \div (bc).$$

61. Not only may a succession of divisions be performed in any order, but a succession of divisions and multiplications may be performed in any order.

For example $$a \times b \div c = a \div c \times b.$$

For $$a = a \div c \times c;$$

$$\therefore \quad a \times b = a \div c \times c \times b$$

$$= a \div c \times b \times c; \text{ [by Art. 39]}$$

therefore, dividing each by c, we have

$$a \times b \div c = a \div c \times b.$$

Hence *we get the same result if we divide the product of two quantities by a third, as if we divide one of the quantities by the third and then multiply by the other.*

62. The operation of division is often indicated by placing the dividend over the divisor with a line between them: thus $\frac{a}{b}$ or a/b means $a \div b$.

When $a \div b$ is written in the fractional form $\frac{a}{b}$, a is called the *numerator* and b the *denominator*.

By putting $a = 1$ in $a \times b \div c = a \div c \times b$,

we have $1 \times b \div c = 1 \div c \times b$,

that is $b \div c = \frac{1}{c} \times b = b \times \frac{1}{c}$,

so that *to divide by any quantity c gives the same result as to multiply by* $\frac{1}{c}$.

63. Since $a^3 \times a^2 = a^5$, and $a^7 \times a^3 = a^{10}$; we have conversely $a^5 \div a^2 = a^3 = a^{5-2}$, and $a^{10} \div a^3 = a^{10-3} = a^7$, and similarly in other cases.

Hence if one power of a letter be divided by another power of the *same* letter, the *index of the quotient is equal to the difference of the indices of the dividend and the divisor.*

64. We have proved in Art. 38, that $a \times (-b) = -ab$;

$$\therefore \quad (-ab) \div (-b) = a, \text{ and } (-ab) \div a = -b.$$

We have also proved that

$$(-a) \times (-b) = +ab, \text{ and } (+a) \times (+b) = +ab;$$

$$\therefore \quad (+ab) \div (-a) = -b \text{ and } (+ab) \div (+a) = +b.$$

Hence if the signs of the dividend and divisor are alike, the sign of the quotient is +; and if the signs of the dividend and divisor are unlike, the sign of the quotient is – ; we therefore have the same **Law of signs** in division as in multiplication.

65. **Division of monomial expressions.** The results arrived at in the preceding articles will enable us to divide any one monomial expression by another. These results are :—

(i) The sign of the quotient is + when the signs of the dividend and divisor are alike, and the sign of the quotient is – when the signs of the dividend and divisor are different.

(ii) The operations of multiplication and division may be performed in any order.

(iii) When one power of any letter is divided by any smaller power of the same letter the index of the quotient is equal to the difference of the indices of the dividend and the divisor.

Ex. 1. Divide $18a^4b^5$ by $6a^2b^3$.

$$\begin{aligned} 18a^4b^5 \div 6a^2b^3 &= 18 \div 6 \times a^4 \div a^2 \times b^5 \div b^3, && \text{from (ii)},\\ &= 3a^{4-2}\,b^{5-3}, && \text{from (iii)}\\ &= 3a^2b^2. \end{aligned}$$

Ex. 2. Divide $15a^3b^6$ by $-5ab^5$.

$$\begin{aligned} 15a^3b^6 \div (-5ab^5) &= -15 \div 5 \times a^3 \div a \times b^6 \div b^5, && \text{from (i) and (ii)},\\ &= -3a^{3-1}\,b^{6-5}, && \text{from (iii)},\\ &= -3a^2b. \end{aligned}$$

Ex. 3. Divide $-5a^7b^5c^4$ by $-7a^3b^2c^4$.

The whole work of dividing one monomial expression by another can be performed *mentally* and the result written down at once. First put down the sign of the result; then divide the numerical coefficient of the dividend by that of the divisor; then take each letter which occurs in the dividend to a power whose index is the difference of the indices of that letter in the dividend and the divisor.

Thus $(-5a^7b^5c^4)\div(-7a^3b^2c^4)=+\frac{5}{7}a^4b^3$, since $c^4\div c^4=1$.

If, in the above example, we had used the rule for finding the index of the quotient of $c^4\div c^4$ we should have been led to the *at present* meaningless result c^0: but $c^4\div c^4$ is obviously 1.

66. **Division of a multinomial expression by a monomial.** When a multinomial expression is divided by a monomial the quotient is equal to the sum of the quotients obtained by dividing its separate terms by that monomial.

Thus $\quad (a+b+\ldots)\div x=a\div x+b\div x+\ldots$

To prove this, multiply by x; then

$$(a+b+\ldots)\div x\times x=a+b+\ldots;$$

also $(a\div x+b\div x+\ldots)\times x=a\div x\times x+b\div x\times x+\ldots$

[Art. 44].

$$=a+b+\ldots$$

Hence $(a+b+\ldots)\div x=a\div x+b\div x+\ldots$

Ex. 1. Divide a^3+2a^2 by a.

The result is $a^3\div a+2a^2\div a=a^2+2a$.

Ex. 2. Divide a^2x^2-3ax by ax.

The result is $a^2x^2\div ax+(-3ax)\div ax=ax-3$.

Ex. 3. Divide $12x^3-5ax^2-2a^2x$ by $3x$.

The result is $12x^3\div 3x+(-5ax^2)\div 3x+(-2a^2x)\div 3x$

$$=4x^2-\tfrac{5}{3}ax-\tfrac{2}{3}a^2.$$

EXAMPLES. XIII.

Divide

1. $10a$ by $-5a$.
2. $-10b$ by $2b$.
3. $-2x$ by $-3x$.
4. a^2 by $-a$.
5. $8ab$ by $-2b$.
6. $-4ab$ by $3a$.
7. $12a^2b^3$ by $-ab^2$.
8. $-6x^2y^3$ by $-4xy$.
9. $-3x^4y^3$ by $2xy^2$.
10. $25a^7b^6c^4$ by $-5a^6bc^4$.
11. $-27a^9b^{11}c^4$ by $-6a^9b^3$.
12. $abcd$ by $-ab$.
13. $a^2b^3c^4d^7$ by $-2ab^2cd^5$.
14. $8a^4x^5y^7z$ by $6ax^3y^4$.
15. $-3a^2b^6cx^7y^9$ by $-2abcx^3y^5$.
16. $3x^2-5ax$ by x.
17. $5y^4-6y^3$ by $-y^2$.
18. $4a^6-5a^5+2a^4$ by a^4.
19. $12a^3+9a^4-6a^5\div -3a^2$.
20. $15a^4b^6-7a^3b^7+9a^2b^4$ by $-3a^2b^3$.

67. **Division by a multinomial expression.** We have now to consider the most general case of division, namely the division of one multinomial expression by another.

The object of division is to find what algebraical expression the divisor must be multiplied by to produce the dividend.

Consider for example the division of

$$8a^3 + 8a^2b + 4ab^2 + b^3 \text{ by } 2a + b.$$

The dividend and the divisor must first be both arranged according to descending (or ascending) powers of some common letter, and the terms of the quotient will be found separately in the same order. In the present case the arrangement is according to descending powers of the letter a. Now the term of highest degree in a in the dividend, namely $8a^3$, must be the product of the terms of highest degree in a in the divisor and the quotient; hence the *first term* of the quotient must be $8a^3 \div 2a = 4a^2$. Now multiply the divisor by $4a^2$ and subtract from the dividend: we then have the remainder $4a^2b + 4ab^2 + b^3$. This remainder must be the product of the divisor by the other terms of the quotient; and hence the first term of the remainder, namely $4a^2b$, must be the product of the first term of the divisor and the next term of the quotient. Hence the *second term* of the quotient is $4a^2b \div 2a = 2ab$. Now multiply the divisor by $2ab$ and subtract the product from the remainder: we then get the second remainder $2ab^2 + b^3$. The *third term* of the quotient is similarly $2ab^2 \div 2a = b^2$. Multiply the divisor by b^2 and subtract the product from $2ab^2 + b^3$, and there is no remainder. Since there is no remainder after the last subtraction, the dividend must be equal to the sum of the different quantities which have been subtracted from it; but we have subtracted in succession the divisor multiplied by $4a^2$, by $2ab$ and by b^2; we have therefore subtracted altogether the product of the divisor and $4a^2+2ab+b^2$. Hence the required quotient is $4a^2+2ab+b^2$.

The process is written in the following form.

$$\begin{array}{rrrrrl}
2a+b\,) & 8a^3 & +\,8a^2b & +\,4ab^2 & +\,b^3 & (\,4a^2+2ab+b^2 \\
 & \underline{8a^3} & \underline{+\,4a^2b} & & & \\
 & & 4a^2b & +\,4ab^2 & +\,b^3 & \\
 & & \underline{4a^2b} & \underline{+\,2ab^2} & & \\
 & & & 2ab^2 & +\,b^3 & \\
 & & & \underline{2ab^2} & \underline{+\,b^3} &
\end{array}$$

From the above example it will be seen that in order to divide one multinomial expression by another we proceed as follows :—

1. *Arrange both dividend and divisor according to ascending or both according to descending powers of some common letter.*

2. *Divide the first term of the dividend by the first term of the divisor: this will give the first term of the quotient.*

3. *Multiply the divisor by the first term of the quotient found above, and subtract the product from the dividend.*

4. *Now treat the remainder as a new dividend, and go on repeating the process.*

68. The following are additional examples:

Ex. 1. Divide $3x^3-4x^2+2x-1$ by $1-x$.

The order of the terms in the divisor must first be changed to $-x+1$.

$$\begin{array}{rrrrrl}
-x+1\,) & 3x^3 & -\,4x^2 & +\,2x & -\,1 & (\,-3x^2+x-1 \\
 & \underline{3x^3} & \underline{-\,3x^2} & & & \\
 & & -\,x^2 & +\,2x & -\,1 & \\
 & & \underline{-\,x^2} & \underline{+\,x} & & \\
 & & & x & -\,1 & \\
 & & & \underline{x} & \underline{-\,1} &
\end{array}$$

Ex. 2. Divide $a^4-a^3b+2a^2b^2-ab^3+b^4$ by a^2+b^2.

$$\begin{array}{r|rrrrrl}
a^2+b^2) & a^4 & -a^3b & +2a^2b^2 & -ab^3 & +b^4 & (a^2-ab+b^2 \\
 & a^4 & & +a^2b^2 & & & \\ \cline{2-6}
 & & -a^3b & +a^2b^2 & -ab^3 & +b^4 & \\
 & & -a^3b & & -ab^3 & & \\ \cline{3-6}
 & & & +a^2b^2 & & +b^4 & \\
 & & & +a^2b^2 & & +b^4 & \\ \cline{4-6}
\end{array}$$

Ex. 3. Divide $a^4+a^2b^2+b^4$ by a^2-ab+b^2.

$$\begin{array}{r|rrrrrl}
a^2-ab+b^2) & a^4 & & +a^2b^2 & & +b^4 & (a^2+ab+b^2 \\
 & a^4 & -a^3b & +a^2b^2 & & & \\ \cline{2-6}
 & & +a^3b & & & +b^4 & \\
 & & +a^3b & -a^2b^2 & +ab^3 & & \\ \cline{3-6}
 & & & +a^2b^2 & -ab^3 & +b^4 & \\
 & & & +a^2b^2 & -ab^3 & +b^4 & \\ \cline{4-6}
\end{array}$$

In the last example the terms of the dividend were placed apart in order that 'like' terms might be placed under one another without altering the order of the terms according to descending powers of a. The subtractions can however be easily performed without putting 'like' terms under one another; but *the arrangement of the terms according to descending (or ascending) powers of the chosen letter should never be departed from.*

69. The following formulae are very important, and should be remembered; they can easily be verified.

$$\begin{aligned}
(x^2+2ax+a^2) &\div (x+a) = x+a, \\
(x^2-2ax+a^2) &\div (x-a) = x-a, \\
(x^2-a^2) &\div (x-a) = x+a, \\
(x^3-a^3) &\div (x-a) = x^2+ax+a^2, \\
(x^3+a^3) &\div (x+a) = x^2-ax+a^2, \\
(x^4-a^4) &\div (x-a) = x^3+ax^2+a^2x+a^3, \\
(x^4-a^4) &\div (x+a) = x^3-ax^2+a^2x-a^3.
\end{aligned}$$

EXAMPLES. XIV.

Divide

1. x^2-5x+6 by $x-2$.
2. $x^2+5x-24$ by $x+8$.
3. $x^2-23x+132$ by $x-11$.
4. $x^2-4x-77$ by $x+7$.
5. $3x^2-4x-4$ by $x-2$.
6. $3x^2-11x+10$ by $3x-5$.
7. a^2-b^2 by $a-b$.
8. x^2-9y^2 by $x-3y$.
9. x^2-16y^2 by $x+4y$.
10. $9x^2-64y^4$ by $3x+8y^2$.
11. $x^2-\frac{1}{4}y^4$ by $x-\frac{1}{2}y^2$.
12. $\frac{1}{9}x^6-\frac{1}{16}a^6$ by $\frac{1}{3}x^3-\frac{1}{4}a^3$.
13. $3x^2+8x+4$ by $6x+12$.
14. $x^2-\frac{7}{12}x-1$ by $4x+3$.
15. a^3-b^3 by $a-b$.
16. $8a^3+27b^3$ by $2a+3b$.
17. $a^3x^3+b^3y^3$ by $ax+by$.
18. $8a^6x^6-125y^6$ by $2a^2x^2-5y^2$.
19. x^3-8x-3 by $3-x$.
20. $2x^3-5x^2+4$ by $2-x$.
21. $2x^4-9x^3+10x^2-5x+6$ by $3-x$.
22. x^4+2x^2-x+2 by $1-x+x^2$.

70. We now give examples which require greater care in the arrangement of the terms.

Ex. 1. Divide $2a^2-2b^2-3c^2-5bc-5ca-3ab$ by $a-2b-3c$.

Where, as in this example, more than two letters are involved, we must not merely arrange the terms according to powers of a, but b also is given the precedence over c. The terms are therefore arranged as under.

$$\begin{array}{rlllllll}
a-2b-3c) & 2a^2 & -3ab & -5ac & -2b^2 & -5bc & -3c^2 & (2a+b+c \\
 & 2a^2 & -4ab & -6ac & & & & \\
\cline{2-7}
 & & ab & +ac & -2b^2 & -5bc & -3c^2 & \\
 & & ab & & -2b^2 & -3bc & & \\
\cline{3-7}
 & & & +ac & & -2bc & -3c^2 & \\
 & & & +ac & & -2bc & -3c^2 & \\
\cline{4-7}
\end{array}$$

Ex. 2. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.

First arrange the dividend according to powers of a, and give b precedence over c throughout.

$$
\begin{array}{l}
a+b+c\,)\,a^3-3abc+b^3+c^3\,(\,a^2-ab-ac+b^2-bc+c^2 \\
\qquad\quad \underline{a^3+a^2b+a^2c} \\
\qquad\qquad -a^2b-a^2c-3abc+b^3+c^3 \\
\qquad\qquad \underline{-a^2b-ab^2-abc} \\
\qquad\qquad\qquad\quad -a^2c+ab^2-2abc+b^3+c^3 \\
\qquad\qquad\qquad\quad \underline{-a^2c\qquad\quad -abc-ac^2} \\
\qquad\qquad\qquad\qquad\quad +ab^2-abc+ac^2+b^3+c^3 \\
\qquad\qquad\qquad\qquad\quad \underline{+ab^2\qquad\qquad\quad +b^3+b^2c} \\
\qquad\qquad\qquad\qquad\qquad\quad -abc+ac^2-b^2c+c^3 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{-abc\qquad\quad -b^2c-bc^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad ac^2+bc^2+c^3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{ac^2+bc^2+c^3}
\end{array}
$$

By using brackets, the above process can be shortened. Thus

$$
\begin{array}{l}
a+b+c\,)\,a^3-3abc+b^3+c^3\,(\,a^2-a\,(b+c)+(b^2-bc+c^2) \\
\qquad\quad \underline{a^3+a^2\,(b+c)} \\
\qquad\qquad -a^2\,(b+c)-3abc+b^3+c^3 \\
\qquad\qquad \underline{-a^2\,(b+c)-a\,(b+c)^2} \\
\qquad\qquad\qquad\qquad a\,(b^2-bc+c^2)+b^3+c^3 \\
\qquad\qquad\qquad\qquad \underline{a\,(b^2-bc+c^2)+(b+c)\,(b^2-bc+c^2)}
\end{array}
$$

Ex. 3. Divide $a^2\,(b-c)+b^2\,(c-a)+c^2\,(a-b)$ by $a-b$.

First arrange the dividend according to descending powers of a. Then

$$
\begin{array}{l}
a-b\,)\,a^2\,(b-c)-a\,(b^2-c^2)+b^2c-bc^2\,(\,a\,(b-c)-c\,(b-c) \\
\qquad \underline{a^2\,(b-c)-ab\,(b-c)} \\
\qquad\qquad\quad -a\,(bc-c^2)+b^2c-bc^2 \\
\qquad\qquad\quad \underline{-a\,(bc-c^2)+bc\,(b-c)}
\end{array}
$$

71. In the process of division as described in Art. 67, the dividend and divisor being arranged according to descending powers of a, we subtract from the dividend at the first subtraction an expression such that the term in it which contains the highest power of a is the same as the term in the dividend which contains the highest power of a; hence it follows that the highest power of a contained

in the remainder after the first subtraction will be *less* than the highest power of a contained in the dividend. Similarly, the highest power of a contained in every remainder will be less than in the preceding remainder; and hence by proceeding far enough we must come to a stage where there is no remainder, or else where there is a remainder the highest power of a in which is less than the highest power of a in the divisor, and in this latter case the division cannot be *exactly* performed.

It is convenient to extend the definition of division to the following: *To divide* A *by* B *is to find an algebraical expression* C *such that* B × C *is either equal to* A, *or differs from* A *by an expression which is of lower degree, in some particular letter, than the divisor* B.

For example, if we divide $x^2+3ax+a^2$ by $x+a$, we have

$$\begin{array}{rl}
x+a\ ) & x^2+3ax+a^2\ (\ x+2a \\
& x^2+\ ax \\ \cline{2-2}
& \qquad 2ax+\ a^2 \\
& \qquad 2ax+2a^2 \\ \cline{2-2}
& \qquad\qquad\quad -\ a^2 \\ \cline{2-2}
\end{array}$$

Thus $(x^2+3ax+a^2)\div(x+a)=x+2a$, with remainder $-a^2$.

We may express the above in the following form:

$$x^2+3ax+a^2=(x+2a)\,(x+a)-a^2.$$

We have also, by arranging the dividend and the divisor differently,

$$\begin{array}{rl}
a+x\ ) & a^2+3ax+x^2\ (\ a+2x \\
& a^2+\ ax \\ \cline{2-2}
& \qquad 2ax+\ x^2 \\
& \qquad 2ax+2x^2 \\ \cline{2-2}
& \qquad\qquad\quad -\ x^2 \\ \cline{2-2}
\end{array}$$

Hence, *when there is a remainder*, a change in the order of the dividend and the divisor leads to a result of different form. This is, however, what might be expected considering that in the first case we find what the divisor must be multiplied by in order to agree with the dividend so far as the terms which contain x are concerned; and in the second case we find what the divisor must be multiplied by, in order to agree with the dividend so far as the terms which contain a are concerned.

When therefore we have to divide one expression by another, both expressions being arranged in the same way, it must be understood that this arrangement is to be adhered to.

EXAMPLES. XV.

Divide

1. x^4+x^2+1 by x^2+x+1. **2.** $1+x^4+x^8$ by $1-x^2+x^4$.

3. x^4+4x^2+16 by x^2+2x+4. **4.** $16x^4+36x^2+81$ by $4x^2-6x+9$.

5. x^4+2x^2+x+2 by x^2+x+1. **6.** $2x^4-x^3+2x^2+1$ by $1-x+x^2$.

7. x^5-4x^3+2x-4 by $2-x$. **8.** x^6-3x^2-52 by $2-x$.

9. $1-x+5x^3-3x^4$ by $1-2x+3x^2$.

10. $x^6+2x^5-4x^4-2x^3+12x^2-2x-1$ by x^2+2x-1.

11. $1-4x^3+3x^4$ by $(1-x)^2$. **12.** $1-5x^4+4x^5$ by $(x-1)^2$.

13. $35+4x-16x^2+19x^3-6x^4$ by $7+5x-3x^2$.

14. $3x^5-12x^4+17x^3-9x^2+1$ by $(1-x)^3$.

15. x^5-5x^2+5x-1 by $1+x^2-2x$.

16. $16x^6-97x^4-84x^2+77x+8$ by $4x^2+11x+1$.

17. $1+2x^5+x^6+2x^7$ by $1+x+x^2$. **18.** $x^{10}+x^5+1$ by x^2+x+1.

19. $6x^5-7x^4y+x^3y^2+20x^2y^3-22xy^4+8y^5$ by $2x^2-3xy+4y^2$.

20. $8y^5-22xy^4+20x^2y^3+x^3y^2-7x^4y+6x^5$ by $4y^2-3xy+2x^2$.

21. $x^2-y^2-xz+yz$ by $x-y$. **22.** $x^2-y^2+2xz-2yz$ by $x-y$.

23. $a^2-b^2+ac-bc$ by $a+b+c$. **24.** $x^2-y^2-z^2+2yz$ by $y-z-x$.

25. $a^2-4b^2-c^2-4bc$ by $2b+c-a$.

26. $a^2+4b^2+9c^2-12bc-6ca+4ab$ by $3c-2b-a$.

27. $2a^2-2b^2-2c^2-4bc+3ca+3ab$ by $a+2b+2c$.

28. $3a^2-16b^2-4c^2+20bc-11ca+8ab$ by $a+4b-4c$.

29. $x^3+y^3-z^3+3xyz$ by $x+y-z$.

30. $8x^3-y^3+z^3+6xyz$ by $y-z-2x$.

31. $27a^3-8b^3+c^3+18abc$ by $3a-2b+c$.

32. $a^3+3a^2b+3ab^2+b^3+c^3$ by $a+b+c$.

33. $a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2$ by $a+b+c$.

34. $(x+1)^3+y^3$ by $x+y+1$. 35. $(x^2-yz)^3+8y^3z^3$ by x^2+yz.

36. $(x^2-2yz)^3-27y^3z^3$ by x^2-5yz.

37. $9a^2+6ab+b^2-4c^2-4cd-d^2$ by $3a+b-2c-d$.

38. $x^3+(4ab-b^2)x-(a-2b)(a^2+3b^2)$ by $x-a+2b$.

39. $a^3(b-c)+b^3(c-a)+c^3(a-b)$ by $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

40. $a^4(b-c)+b^4(c-a)+c^4(a-b)$ by $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

MISCELLANEOUS EXAMPLES. I.

A. 1. Find the value of $(y-z)^2+(z-x)^2+(x-y)^2$ when $x=-1$, $y=0$ and $z=1$.

2. Add together $3a^2-2ca-2ab$, $2b^2+3bc+3ba$ and $c^2-2ac-2bc$.

3. Shew that

$$a^3+b^3+c^3-3abc-a(a^2-bc)-b(b^2-ca)-c(c^2-ab)=0.$$

4. Multiply $2x^2-\frac{1}{2}x+\frac{1}{3}$ by $\frac{1}{2}x+3$,

and $\frac{1}{4}x^2+\frac{1}{2}xy+y^2$ by $\frac{1}{4}x^2-\frac{1}{2}xy+y^2$.

5. Simplify $(x+1)(x+2)(x+3)-(x-1)(x-2)(x-3)$.

6. Divide $2-12x^5+10x^6$ by $1-2x+x^2$.

B. 1. Find the value of $c^2+b(c+a)$, and of $\sqrt{2bc}-a$, when $a=11$, $b=3$ and $c=6$.

2. Subtract $5a^4-3a^3b+4a^2b^2$ from $5b^4-3b^3a+4b^2a^2$.

3. Simplify $x-(1-\overline{1-x})$, $3x-\overline{7-4x}$

and $b-2a-\{c-a-(b-a+c)\}$.

4. Multiply $9x^2-1$ by $x^2+\frac{1}{9}$, and $a+b+c$ by $a+b-c$.

5. Shew that

$$2(a^2+b^2+c^2-bc-ca-ab)=(b-c)^2+(c-a)^2+(a-b)^2.$$

6. Divide $x^3-3x^2+3x+y^3-1$ by $x+y-1$.

C. 1. Find the numerical value of

$$\frac{\sqrt{a^2+2bc}}{a}+\frac{\sqrt{b^2+ca}}{b}+\frac{\sqrt{c^2+ab}}{c},$$

when $a=4$, $b=3$ and $c=-2$.

2. Add together $3a^2b-5ab^2+7b^3$ and $2a^3-\frac{1}{2}a^2b+5ab^2-\frac{4}{5}b^3$.

3. Multiply $ax^2 - x + a$ by $ax^2 + x + a$, and find the square of

$$2a + b - 3c.$$

4. Simplify $(x+y)^2 - (x+y)(x-y) - \{x(2y-x) - y(2x-y)\}$.

5. The product of two algebraical expressions is

$$x^6 + x^5y + x^4y^2 - x^3y^3 + y^6,$$

and one of them is $x^2 + xy + y^2$; what is the other?

6. Prove

(i) $a(b-c) + b(c-a) + c(a-b) = 0$,

(ii) $a^2(b-c) + b^2(c-a) + c^2(a-b) + (b-c)(c-a)(a-b) = 0$.

D. 1. If $a=1$, $b=2$ and $c=-3$, find the value of $6a+3b+4c$, and of $a^3+b^3+c^3-3abc$.

2. Simplify $3\{x - 2(y-z)\} - [4y + \{2y - (z-x)\}]$.

3. What must be added to $(a+b+c)^2$ that the sum may be $(a-b-c)^2$?

4. Multiply a^3+b^3 by $a-b$, and divide the result by $a+b$.

5. Divide x^3+z^3 by $x+z$, and from the result write down the quotient when $(x+y)^3+z^3$ is divided by $x+y+z$.

6. Prove

(i) $(x+y)(x-y) + (y+z)(y-z) + (z+x)(z-x) = 0$,

(ii) $(x-y)^2 + (y-z)^2 + (z-x)^2$

$$= 2(x-y)(x-z) + 2(y-z)(y-x) + 2(z-x)(z-y).$$

E. 1. Find the value of

$$\frac{2\sqrt{(a^2+b^2)} + \sqrt[3]{(a^2+b^2-c^2+6d^2)}}{d-c+b-a},$$

when $a=3$, $b=4$, $c=5$ and $d=6$.

2. Simplify $4x - (y-x) - 3\{2y - 3(x+y)\}$ and

$$2x - 3y - 4(x-2y) + 5\{3x - 2(x-y)\},$$

and multiply the two results.

3. Shew that $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$, and verify the result when $a=1$ and $b=-2$.

4. Find the coefficient of x^2 in the product of $x^2 - (a-b)x - ab$ and $x^2 + (a+b)x + ab$.

5. Divide the difference of $(2a+3b)^2$ and $(3a+2b)^2$ by $a+b$; divide also the sum of $(2a-3b)^3$ and $(3a-2b)^3$ by $a-b$.

6. Shew that

$$(a+b+c-d)\,(a+b-c+d)+(a-b+c+d)\,(-a+b+c+d)=4\,(ab+cd).$$

F. 1. Find the value of

$$\frac{(a+b)\,(c+d)-(b+c)\,(d+a)}{ab+bc-cd-da},$$

when $a=3$, $b=4$, $c=5$ and $d=-4$.

2. Find the value of $(a-x)\,(a-\frac{1}{2}y)\,(a-\frac{1}{4}z)$.

3. If x^2+7x+c is exactly divisible by $x+4$, what is the value of c?

4. Divide $x^2-y^2-z^2+2yz$ by $x+y-z$, and find the coefficient of x in the quotient obtained by dividing $8x^4+xy^3-y^4$ by $x-\frac{1}{2}y$.

5. Shew that

$$(a^2+b^2)\,(c^2+d^2)=(ac+bd)^2+(ad-bc)^2=(ac-bd)^2+(ad+bc)^2.$$

6. Shew that

$$(b-c)^2+(a-b)\,(a-c)=(c-a)^2+(b-c)\,(b-a)=(a-b)^2+(c-a)\,(c-b).$$

G. 1. Simplify $5x-3\,[2x+9y-2\,\{3x-4\,(y-x)\}]$.

2. Subtract $6a^2-4ab+7b^2$ from $2a^2-4ab+2b^2$, and from the remainder subtract $4a^2-6ab+5b^2$.

3. Multiply $a^2+b^2+1-ab-a-b$ by $a+b+1$.

4. Divide $a^3+b^3-8c^3+3a^2b+3ab^2$ by $a+b-2c$.

5. The product of two algebraical expressions is x^7-64x, and one of the expressions is x^2-2x+4: what is the other?

6. Prove that

$$(1+x+x^2)\,(1-x+x^2)\,(1-x^2+x^4)\,(1-x^4+x^8)=1+x^8+x^{16},$$

and that

$$a^3\,(b-c)+b^3\,(c-a)+c^3\,(a-b)+(b-c)\,(c-a)\,(a-b)\,(a+b+c)=0.$$

Simple Equations

CHAPTER V.

SIMPLE EQUATIONS.

72. A STATEMENT of the equality of two algebraical expressions is called an **equation**, and the two equal expressions are called the **members** or **sides** of the equation.

When the statement of equality is true for *all values* of the letters involved, the equation is sometimes called an **identical equation.** An identical equation is however generally called an **identity**, and the name *equation* is reserved for those cases in which the equality is only true for certain particular values of the letters involved.

> Thus $a+a=2a$, and $(a+b)^2=a^2+2ab+b^2$, which are true for *all values* of a and b, are *identities;* and $5a+2=12$, which is only true when a is 2, is an *equation.*

Note.—For the sake of distinction, a quantity which is supposed to be known, but which is not expressed by any particular arithmetical number, is represented by one of the first letters of the alphabet a, b, c, &c., and a quantity which is unknown, and which is to be found, is represented by one of the last letters of the alphabet x, y or z.

73. To **solve an equation** is to find the value, or values, of the unknown quantity for which the equation is true; and these values of the unknown quantity are said to **satisfy** the equation, and are called the **roots** of the equation.

74. An equation which contains only one unknown quantity, x suppose, is said to be of the **first degree** when x occurs only in the first power; it is said to be of the **second degree** when x^2 is the highest power of x which occurs, and so on.

Equations of the first degree are however generally called **simple equations**, and equations of the second degree are generally called **quadratic equations.**

In the present chapter we shall only consider simple equations which contain one unknown quantity.

75. In the solution of equations frequent use is made of the following **axioms**:—

(i) If we add to equals the same quantity, or equal quantities, the sums will be equal.

Thus, if $a=b$, then $a+c=b+c$.

(ii) If we take from equals the same quantity, or equal quantities, the differences will be equal.

Thus, if $a=b$, then $a-c=b-c$.

(iii) If we multiply equals by the same quantity, or by equal quantities, the products will be equal.

Thus, if $a=b$, then $ac=bc$.

(iv) If we divide equals by the same quantity not zero, or by equal quantities, the quotients will be equal.

Thus, if $a=b$, then $a \div c = b \div c$. But although $2 \times 0 = 3 \times 0$, we cannot divide by 0 and say that $2=3$.

76. Let $$a+b=c-d.$$

Add $-b$ to both sides, then the equality still holds good by Axiom (ii);

$$\therefore\ a+b-b=c-d-b,$$

that is $$a=c-d-b.$$

Thus the term b has been cancelled from one side of the equation, and it appears on the other side with its sign changed from + to −.

Again, add d to both sides of the last equation; then, by Axiom (i),

$$a+d=c-d-b+d;$$

$$\therefore\quad a+d=c-b.$$

Thus the term d has been cancelled from one side of the equation, and it appears on the other side with its sign changed from − to +.

We can proceed in a similar manner in any other case; hence *any term may be moved from one side of an equation to the other, provided its sign is changed.*

When terms are changed from one side of an equation to the other side, they are said to be **transposed.**

77. We may change the signs of *all* the terms of an equation; for, by Axiom (iii), we do not destroy the equality by multiplying both sides, and therefore every term, by -1; and this multiplication will change the sign of every term.

78. The method of solving simple equations will be seen from the following examples:—

Ex. 1. Solve $8x+7=4x+27.$

Transpose the terms $4x$ and 7, then

$$8x-4x=27-7.$$

Combine like terms, then

$$4x=20.$$

Divide both sides by 4, the coefficient of x; then

$$x=5.$$

Ex. 2. Solve $5x-7=7x-15.$

By transposition, we have

$$5x-7x=-15+7.$$

Combining like terms, we get

$$-2x=-8.$$

Divide both sides by -2, the coefficient of x; then

$$x=4.$$

Ex. 3. Solve $\frac{x}{2}+2=\frac{x}{4}+\frac{5}{2}$.

Multiply every term by 4, the L.C.M. of the denominators, then the fractions will be got rid of, and we shall have

$$2x+8=x+10.$$

Transposing

$$2x-x=10-8;$$

$$\therefore \quad x=2.$$

Ex. 4. Solve $\frac{x+1}{4}-\frac{1}{3}(x-1)=1$.

Multiply by 12 to get rid of fractions; then

$$\tfrac{12}{4}(x+1)-\tfrac{12}{3}(x-1)=12,$$

that is $3(x+1)-4(x-1)=12$;

$$\therefore \quad 3x+3-4x+4=12.$$

Transposing

$$3x-4x=12-3-4;$$

$$\therefore \quad -x=5,$$

or, by changing the signs,

$$x=-5.$$

Ex. 5. Solve $ax+b^2=bx+a^2$.

By transposition,

$$ax-bx=a^2-b^2,$$

that is $(a-b)x=a^2-b^2$.

Divide by $a-b$, the coefficient of x; then

$$x=(a^2-b^2)\div(a-b)=a+b.$$

Note. The student should test his results by seeing that the values obtained really *satisfy* the given equations.

For example, if we put 5 for x in Ex. 1, we have

$$8\times5+7=4\times5+27,$$

that is $$40+7=20+27,$$ which is clearly true.

Again, if we put $a+b$ for x in Ex. 5, we have

$$a(a+b)+b^2=b(a+b)+a^2,$$

that is $$a^2+ab+b^2=ab+b^2+a^2,$$

which is an identity.

79. From the above it will be seen that the different steps in the process of solving a simple equation are as follows. First, clear the equation of fractions, and perform all the algebraical operations which are indicated. Then transpose all the terms into which the unknown quantity enters to one side of the equation, and all the other terms to the other side. Next combine all the terms which contain the unknown quantity into one term, and divide by the coefficient of the unknown quantity: this gives the required root.

Note. In order to acquire a habit of clear and accurate thought, the way in which each new equation is derived from the preceding should always be indicated.

A beginner should be cautioned against the common mistake of putting a meaningless sign of equality at the commencement of a line.

80. The following are additional examples of simple equations.

Ex. 1. Solve $(x-1)(x-2)+5=(x+1)^2$.

Removing the brackets, we have

$$x^2-3x+2+5=x^2+2x+1.$$

Transposing, we have

$$x^2-3x-x^2-2x=1-2-5$$

$$\therefore \qquad -5x=-6.$$

Divide by -5; then

$$x=\tfrac{6}{5}.$$

Ex. 2. Solve $3(x-1)-\{3x-(2-x)\}=5$.

Removing the brackets, we have

$$3x-3-3x+2-x=5;$$

$$\therefore \quad 3x-3x-x=5+3-2;$$

$$\therefore \quad -x=6,$$

or

$$x=-6.$$

Ex. 3. Solve $3x^2-1=(3x+2)(x-5)$.

Removing the brackets, we have

$$3x^2-1=3x^2-13x-10.$$

Transposing, we have

$$3x^2-3x^2+13x=-10+1.$$

Hence

$$13x=-9,$$

$$\therefore \quad x=-\tfrac{9}{13}.$$

Ex. 4. Solve $a(x-a)+b(x-b)=2ab$.

Removing the brackets, we have

$$ax-a^2+bx-b^2=2ab.$$

Transposing, we have

$$ax+bx=a^2+b^2+2ab,$$

that is

$$x(a+b)=(a+b)^2;$$

$$\therefore \quad x=(a+b)^2\div(a+b)=a+b.$$

EXAMPLES. XVI.

Solve the following equations :—

1. $3x+4=x+10$.

2. $x+7=4x+4$.

3. $5x-12=6x-8$.

4. $7x+19=5x+7$.

5. $3(x-2)=2(x-3)$.

6. $5(x+2)=3(x+3)+1$.

7. $x-(4-2x)=7(x-1)$.

8. $5(4-3x)=7(3-4x)$.

9. $2(x-3)=5(x+1)+2x-1$.

10. $4(1-x)+3(2+x)=13$.

11. $2(x-2)+3(x-3)+4(x-4)-20=0$.

12. $2(x-1)-3(x-2)+4(x-3)+2=0$.

13. $5x+6(x+1)-7(x+2)-8(x+3)=0$.

14. $x+\frac{2}{3}x=10$.

15. $\dfrac{x}{5}-\dfrac{x}{4}=1$.

16. $\dfrac{x-1}{2}+\dfrac{x-2}{3}=3$.

17. $\frac{1}{4}(x+1)-\frac{2}{3}(x-1)=3$.

18. $\frac{1}{2}(2-x)-\frac{1}{5}(5x+21)=x+3.$

19. $\frac{1}{2}(x-2)+\frac{1}{3}(x-3)+\frac{1}{4}(x-4)=10.$

20. $\frac{x+1}{2}+\frac{x+2}{3}+\frac{x+4}{4}+8=0.$ **21.** $\frac{x-5}{2}-\frac{x-4}{3}=\frac{x-3}{2}-(x-2).$

22. $\frac{x+1}{2}+\frac{x+2}{3}+\frac{x+3}{4}=16.$ **23.** $2x-[3-\{4x+(x-1)\}-5]=8.$

24. $1-2\{x-3(1+x)\}=0.$ **25.** $(x+1)(x+2)=(x-2)(x-4).$

26. $(x-1)(x-2)=(x-3)(x-4).$

27. $2x^2=(x+1)^2+(x+3)^2.$

28. $3x^2=(x+1)^2+(x+2)^2+(x+3)^2.$

29. $(x-2)(x-5)+(x-3)(x-4)=2(x-4)(x-5).$

30. $(x-1)^2+4(x-3)^2=5(x+5)^2.$

31. $5(x+1)^2+7(x+3)^2=12(x+2)^2.$

32. $(x-1)(x-4)=2x+(x-2)(x-3).$

33. $(x-1)^3+(x-2)^3+(x-3)^3=3(x-1)(x-2)(x-3).$

34. $\frac{x+\frac{1}{2}}{2}-\frac{2x-\frac{1}{2}}{5}+1\frac{1}{4}=0.$ **35.** $\frac{3x+5}{8}-\frac{21+x}{2}=5x-15.$

36. $\cdot5x+3\cdot75=5\cdot25x-1.$ **37.** $\cdot25x+4-\cdot375x=\cdot2x-9.$

38. $\cdot15x+1\cdot2-\cdot875x+\cdot375=\cdot0625x.$

39. $1\cdot2x-\frac{1}{5}(\cdot18x-\cdot05)=\cdot4x+8\cdot9.$

40. $a(x-a)=b(x-b).$ **41.** $2(x-a)+3(x-2a)=2a.$

42. $\frac{1}{2}(x+a+b)+\frac{1}{3}(x+a-b)=b.$ **43.** $(a+b)x+(a-b)x=a^2.$

44. $(a+b)x+(b-a)x=b^2.$

45. $\frac{1}{2}(a+x)+\frac{1}{3}(2a+x)+\frac{1}{4}(3a+x)=3a.$

46. $\frac{xa}{b}+\frac{xb}{a}=a^2+b^2.$ **47.** $(a+bx)(b+ax)=ab(x^2-1).$

48. $(a^2+x)(b^2+x)=(ab+x)^2.$ **49.** $a(x+a)+b(b-x)=2ab.$

50. $(x+a+b)^2+(x+a-b)^2=2x^2.$

51. $(x-a)(x-b)+(a+b)^2=(x+a)(x+b).$

52. $(x+a+b+c)(x+a-b-c)=(x-a-b+c)(x-a+b-c).$

53. $ax(x+a)+bx(x+b)=(a+b)(x+a)(x+b).$

54. $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c).$

CHAPTER VI.

PROBLEMS.

81. We shall in the present chapter consider a class of questions called *problems*. In a problem the magnitudes of certain quantities, some of which are known and others unknown, are connected by given relations; and the values of the unknown quantities have to be found by means of these relations.

In order to solve a problem the relations between the known and unknown quantities must be expressed by means of algebraical symbols: we thus obtain *equations*, the roots of which are the required values of the unknown quantities.

82. In the present chapter we shall only consider problems in which there is *one* unknown quantity, and in which the relation between the known and the unknown quantities is expressed algebraically by means of a *simple* equation.

Of such problems the following are examples:

Ex. 1. What number is that whose double exceeds its half by 27?

Let x represent the number. Then the double of the number is $2x$, and its half is $\frac{x}{2}$. By the question $2x$ exceeds $\frac{x}{2}$ by 27.

Hence we have the equation

$$2x = \frac{x}{2} + 27.$$

Transposing $$2x - \frac{x}{2} = 27,$$

that is $$\tfrac{3}{2}x = 27;$$

$$\therefore \quad x = 27 \div \tfrac{3}{2} = 18.$$

Thus 18 is the number required.

Ex. 2. A has £4 and B has 15 shillings. How much must A give to B in order that he may have just four times as much as B?

Let x be the *number of shillings* that A gives to B.

Then A will have $80 - x$ shillings, and B will have $15 + x$ shillings. But A now has four times as much as B. Hence we have the equation

$$80 - x = 4(15 + x).$$

That is $$80 - x = 60 + 4x.$$

Transposing $$-x - 4x = 60 - 80,$$

that is, $$-5x = -20.$$

Divide by -5, then $$x = 4.$$

Thus A must give 4 shillings to B.

Note. It should be remembered that x must always stand for a *number*. It is also to be noticed that in any problem all concrete quantities of the same kind must be expressed in terms of the same unit; for example, in the above all sums of money were expressed as shillings.

Ex. 3. A line 20 inches long is divided into two parts, one of which is double the other. How long are the parts?

Let x be the *number of inches* in the smaller part. Then, since the two parts together make up 20 inches, the number of inches in the larger part is $20 - x$. By the question the larger part is double the smaller; hence we have the equation

$$20 - x = 2x;$$

$$\therefore \quad 20 = 3x,$$

or $$3x = 20.$$

Hence $$x = \tfrac{20}{3} = 6\tfrac{2}{3}.$$

Thus one part is $6\frac{2}{3}$ inches, and the other is $13\frac{1}{3}$ inches.

Ex. 4. A man has 12 coins some of which are half-crowns and the rest shillings, and the coins are worth 18 shillings altogether. How many are there of each kind?

Let x be the number of half-crowns; then $12-x$ will be the number of shillings. The half-crowns are worth $\frac{5}{2}x$ shillings, and the shillings are worth $12-x$ shillings. Hence since the coins are worth 18 shillings altogether, we have the equation

$$\tfrac{5}{2}x+(12-x)=18.$$

Therefore $$\tfrac{5}{2}x-x=18-12,$$

that is $$\tfrac{3}{2}x=6;$$

$$\therefore \quad x=6\div\tfrac{3}{2}=4.$$

Hence there are 4 half-crowns, and 8 shillings.

Ex. 5. A father is six times as old as his son, and in four years he will be four times as old. How old is each?

Let the son be x years old. Then the father must be $6x$ years old. After four years the son will be $x+4$, and the father will be $6x+4$ years old. Hence by the question

$$6x+4=4\,(x+4),$$

that is $$6x+4=4x+16.$$

Hence $$6x-4x=16-4,$$

that is $$2x=12;$$

$$\therefore \quad x=6.$$

Hence the son is 6 years old, and the father is 36 years old.

Ex. 6. *A* can do a piece of work in 12 hours, which *B* can do in 4 hours. *A* begins the work, but after a time *B* takes his place, and the whole work is finished in 6 hours from the beginning. How long did *A* work?

Let $x=$ the number of hours that A worked.

Then $6-x=$ the number of hours that B worked.

Since A can do the whole work in 12 hours, the part done by A in 1 hour is $\frac{1}{12}$.

Therefore the part done by A altogether is $\dfrac{x}{12}$.

Since B can do the whole work in 4 hours, the part done by B in 1 hour is $\frac{1}{4}$.

Therefore the part done by B altogether is $\frac{1}{4}(6-x)$.

But *A* and *B* together do the whole of the work. Hence we have the equation

$$\frac{x}{12}+\tfrac{1}{4}(6-x)=1.$$

Multiply by 12, then

$$x+3(6-x)=12,$$

that is $$x+18-3x=12.$$

Transposing $$-2x=-6;$$

$$\therefore \qquad x=3.$$

Hence *A* worked for 3 hours.

Ex. 7. Find the time between 3 and 4 at which the hands of a clock are together.

Suppose that the hands are together at x minutes after 3 o'clock.

At 3 o'clock the hour-hand is 15 minute-spaces in front of the minute-hand, and after x minutes they are together. Hence while the minute-hand moves through x minute-spaces the hour-hand will move through $x-15$ such spaces. But the minute-hand moves twelve times as fast as the hour-hand, and therefore in any time the minute-hand passes over twelve times as many minute-spaces as the hour-hand.

Hence $$x=12(x-15),$$

that is $$x=12x-180;$$

$$\therefore \qquad 11x=180,$$

or $$x=\tfrac{180}{11}=16\tfrac{4}{11}.$$

Thus the time required is $16\tfrac{4}{11}$ minutes past 3 o'clock.

EXAMPLES. XVII.

1. Find two numbers whose sum is 200, and whose difference is 2.

2. Find two numbers whose sum is 56, and whose difference is 20.

3. Divide 25 into two parts whose difference is 5.

4. Divide 100 into two parts whose difference is 45.

5. What number is that to which if you add 40 the sum will be three times the original number?

6. What number is that from which if you take away 14 the remainder is one-third of the original number?

7. The difference of two numbers is 15, and one number is four times the other. Find the numbers.

8. Find two numbers whose difference is 10, and one of which is three times the other.

9. The sum of two numbers is 38, and one of them exceeds twice the other by 2. What are the numbers?

10. Find two numbers the sum of which is 31, and which are such that one of them is less by 2 than half the other.

11. Find a number whose fourth part exceeds its fifth part by 2.

12. Find a number whose third part exceeds its seventh part by 80.

13. Find a number which when multiplied by 4 is as much above 35 as it was originally below it.

14. Find a number which when multiplied by 8 exceeds 27 as much as 27 exceeds the original number.

15. Four times the difference between the fourth and fifth parts of a certain number exceeds by 4 the difference between the third and seventh parts. What is the number?

16. Fifty times the difference between the seventh and eighth parts of a certain number exceeds half the number by 44. What is the number?

17. Divide 100 into two parts such that three times one of the parts plus five times the other is 410.

18. Divide 100 into two parts such that twice one part is equal to three times the other.

19. The difference of two numbers is 20, and one half of one of the numbers is equal to one-fifth of the other. Find them.

20. The sum of two numbers is 36, and their difference is half the greater. Find them.

21. *A* has £100, and *B* has £20; how much must *A* give *B* in order that *B* may have half as much as *A*?

22. *A* and *B* play for a stake of 5*s*. If *A* loses he will have as much as *B*, but if *A* wins he will have three times as much as *B*. How much has each?

23. *A* and *B* have £50 between them. *A* wins from *B* as much as he had originally, and he then had three times as much as *B*. What had *A* at first?

24. *A*, *B* and *C* have a certain sum between them. *A* has one half of the whole, *B* has one-third of the whole, and *C* has £50. How much have *A* and *B*?

25. *A* and *B* together have £75, and *A* has £5 more than *B*: how much has each?

26. *A* has £5 more than *B*, *B* has £20 more than *C*, and they have £360 between them. How much has each?

27. *A* has £15 more than *B*, *B* has £5 less than *C*, and they have £65 between them. How much has each?

28. *A* has £5 less than *B*, *C* has as much as *A* and *B* together, and they have £100 between them. How much has each?

29. Divide £30 among 10 men, 20 women and 40 children, giving to each man 15 shillings more than to each child, and to each woman as much as to two children.

30. Divide 15 shillings among 3 men, 5 women and 20 children, giving to each man one shilling more than to each woman, and to each child half as much as to each woman.

31. A man of 40 has a son 10 years old; in how many years will the father be three times as old as the son?

32. One man is 70 and another is 50 years of age; when was the first twice as old as the second?

33. A father's age is three times that of his son, and in 10 years it will be twice as great: how old are they?

34. In 5 years a father will be just four times as old as his daughter, and in 10 years he will be just three times as old: how old is he now?

35. A sum of money is divided between three persons *A*, *B* and *C* in such a way that *A* and *B* have £60 between them, *A* and *C* have £65, and *B* and *C* have £75. How much has each?

36. Four persons *A*, *B*, *C*, *D* are possessed of certain sums of money, such that *A* and *B* together have £49, *A* and *C* together have £51, *B* and *C* together have £53, and *A* and *D* together have £47. How much has each?

37. What is the price of beef if a reduction of 20 per cent. in the price would enable a purchaser to obtain 6 lbs. more for a sovereign?

38. How much are eggs a score, when a rise of 20 per cent. in the price would make a difference of 80 in the number which could be bought for a sovereign?

39. A man leaves one half of his property to his wife, one-third to a son, and the remainder (which is £2000) to a daughter. How much did he leave altogether?

40. A man left his property to be divided between his three children in such a way that the share of the eldest was to be twice that of the second, and the share of the second twice that of the youngest. It was found that the eldest received £750 more than the youngest. How much did each receive?

41. A purse contains 28 coins which altogether amount to £7. A certain number of the coins are shillings, one-fifth of that number are half-sovereigns, and the rest are sovereigns. Find the number of each.

42. A purse contains 36 coins which altogether amount to £11. A certain number of the coins are sovereigns, there are three times as many half-sovereigns, and the rest are shillings. Find the number of each.

43. Find the time between 5 o'clock and 6 o'clock when the hands of a watch are together.

44. Find at what time between 9 and 10 o'clock the hands of a watch are at right angles to one another.

45. There are two numbers one of which exceeds the other by 3, while its square exceeds the square of the other by 99. Find the numbers.

46. In a mixture of spirits and water half of the whole plus 25 gallons was spirit; and a third of the whole minus 5 gallons was water: how many gallons were there of each?

47. A garrison of 1000 men having provisions for 60 days was reinforced after 10 days, and from that time the provisions only lasted 20 days. Find the number in the reinforcement.

48. A labourer was engaged for 36 days, upon the condition that he should receive 2*s.* 6*d.* for every day he worked, but should pay 1*s.* 6*d.* for every day he was idle. At the end of the time he received 58*s.* How many days did he work?

49. A sum of money is divided among three persons. The first receives £10 more than a third of the whole sum; the second receives £15 more than a half of what remains; and the third receives what is over, which is £70. Find the original sum.

50. A purse of sovereigns is divided amongst three persons, the first receiving half of them and one more, the second half of the remainder and one more, and the third 6. Find the number of sovereigns the purse contained.

51. From a sum of money £20 more than its half was taken away: from the remainder £30 more than its third part: and from what then remained £40 more than its fifth part; after which there was nothing left. What was the sum?

52. At a cricket match the contractor provided dinner for 24 persons, and fixed the price so as to gain $12\frac{1}{2}$ per cent. upon his outlay. Three of the cricketers were absent. The remaining 21 paid the fixed price for their dinner, and the contractor lost 1*s*. What was the charge for dinner?

53. *A* can do a piece of work in 30 days, which *B* can do in 20 days. *A* begins the work, but after a time *B* takes his place, and the whole work is finished in 25 days from the beginning. How long did *A* work?

54. *A* can do a piece of work in 20 days, which *B* can do in 30 days. *A* begins the work, but after a time *B* takes his place and finishes it; and *B* worked for 10 days longer than *A*. How long did *A* work?

55. How many men are there in a regiment which can be drawn up in two hollow squares with the men 3 and 5 deep respectively, if the one square will just fit within the other, and the same number of men be in each square?

CHAPTER VII.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

83. A SINGLE equation which contains *two* unknown quantities can be satisfied by an indefinite number of pairs of values of the unknown quantities. Consider, for example, the equation $x - 3y = 24$. It is clear that the equation is satisfied by the values $y = 0$ and $x = 24$; or $y = 1$ and $x = 27$; or $y = 2$ and $x = 30$. In fact, we may suppose y to be equal to *any* quantity k, provided that x is taken equal to $24 + 3k$.

If there are two equations containing two unknown quantities, each equation taken by itself can, as we have just seen, be satisfied in an indefinite number of ways; but this is not the case when both equations are to be satisfied by the *same values* of the unknown quantities.

Def. Two or more equations which are to be satisfied by the same values of the unknown quantities contained in them, are called **simultaneous equations.**

84. The *degree* of an equation which contains the two unknown quantities x and y, is the *degree of that term which is of the highest dimensions in x and y.* [Art. 53.]

Similarly the degree of an equation which contains the unknown quantities x, y and z is the degree of that term which is of the highest dimensions in x, y and z.

Thus $4x+5y=20$ is an equation of the *first degree;* and $3x^2-2xy=7$ is an equation of the *second degree.*

The equations $ax^2+by^2+c=0$ and $x+y=xy$ are also of the *second degree.*

85. In the present chapter we shall shew how to solve simultaneous equations of the first degree.

It will be seen that any two simultaneous equations containing two unknown quantities can be solved by deducing from the given equations a third equation from which one of the unknown quantities is absent; and this will give the value of that unknown quantity which is retained.

86. We will shew how to solve two simultaneous equations of the first degree by taking as an example the following pair of equations:

$$3x+5y=22,$$

and $$7x-4y=20.$$

Multiply each member of the first equation by 7, and each member of the second equation by 3: the equations then become

$$21x+35y=154,$$

and $$21x-12y=60.$$

The coefficients of x are now the same in the two equations; hence if we *subtract* each member of the second equation from the corresponding member of the first, we shall obtain an equation from which x is absent: the equation will be

$$47y=94;$$

whence $$y=2.$$

If we put this value of y in the first of the given equations, we have $3x+10=22$; hence $3x=22-10$, or $x=4$.

We may if we please find the value of x before finding the value of y. To do this, multiply the first equation by 4, and the second equation by 5: the equations then become

$$12x+20y=88,$$

and

$$35x-20y=100.$$

The coefficients of y in the two equations are now equal in magnitude but opposite in sign; hence if we *add* each member of the second equation to the corresponding member of the first, we shall obtain an equation from which y is absent: the equation will be

$$47x=188;$$

whence

$$x=4.$$

We can now find the value of y by putting $x=4$ in the first of the given equations.

It will be seen from the above example that in order to solve two simultaneous equations of the first degree we can proceed as follows:—Multiply the given equations by numbers such that in the resulting equations the coefficients of one of the unknown quantities may be equal in magnitude; then by addition or subtraction we shall obtain a *simple* equation which contains only *one* of the unknown quantities, and which can be solved as in the preceding chapter.

The unknown quantity which has been got rid of by the above process is said to have been **eliminated.**

It is generally best to eliminate that unknown quantity which has the smaller coefficients in the two equations.

87. The following are additional examples:

Ex. 1. Solve $3x+2y=13$, $7x+3y=27$.

Multiply the equations by 3 and 2 respectively; we then have

$$9x+6y=39,$$

and

$$14x+6y=54.$$

By subtraction we eliminate y, and have

$$-5x=-15;$$

$$\therefore \quad x=3.$$

Put this value of x in the first of the given equations; then

$$9+2y=13;$$

$$\therefore \quad y=2.$$

Thus $x=3$, $y=2$ are the values required.

Ex. 2. Solve $2x-3y+14=0$, $-4x+5y=26$.

We have $$2x-3y=-14,$$

and $$-4x+5y=26.$$

Multiply the first equation by 2; then

$$4x-6y=-28.$$

From the last two equations we have by addition

$$-y=-2,$$

or $$y=2.$$

We then have from the first equation,

$$2x-6=-14,$$

that is $$2x=-14+6=-8;$$

$$\therefore \quad x=-4.$$

Hence the values required are

$$x=-4, \quad y=2.$$

Ex. 3. Solve $\frac{5x}{3}-\frac{y}{4}=9$, $6x-\frac{7y}{4}=29$.

Clear of fractions by multiplying the equations by 12 and 4 respectively; then

$$20x-3y=108,$$

$$24x-7y=116.$$

Multiply the first equation by 7 and the second by 3; then

$$140x-21y=756,$$

$$72x-21y=348.$$

By subtraction, we have

$$68x=408;$$

$$\therefore \quad x=6.$$

Then $$120-3y=108,$$

whence $$y=4.$$

EXAMPLES. XVIII.

Solve the equations:

1. $7x+4y=1$,
$9x+4y=3$.

2. $3x+5y=19$,
$5x-4y=7$.

3. $x-11y=1$,
$111y-9x=99$.

4. $8x-21y=5$,
$6x+14y=-26$.

5. $34x-15y=4$,
$51x+25y=101$.

6. $39x-15y=93$,
$65x+17y=113$.

7. $19x+85y=350$,
$17x+119y=442$.

8. $8x-11y=0$,
$25x-17y=139$.

9. $3x-11y=0$,
$19x-19y=8$.

10. $\frac{x}{2}+\frac{y}{3}=1$,
$\frac{x}{4}-\frac{2y}{3}=3$.

11. $\frac{x}{3}-\frac{y}{6}=\frac{1}{2}$,
$\frac{x}{5}-\frac{3y}{10}=\frac{1}{2}$.

12. $\frac{x}{3}+3y+14=0$,
$\frac{x}{5}+5y+4=0$.

13. $\frac{x}{5}+5y=-4$,
$\frac{y}{5}+5x=4$.

14. $\frac{x+2}{3}+4y=2$,
$\frac{y+11}{11}-\frac{x+1}{2}=1$.

15. $\frac{2x+3y}{5}+\frac{y+6}{7}=2$,
$\frac{2x-5y}{3}+\frac{x+7}{4}=1$.

16. $4x-\frac{1}{3}(y-3)=5x-3$,
$2y+\frac{1}{3}(2x-5)=\frac{21y+37}{6}$.

17. $\frac{x}{2}-\frac{1}{3}(y-2)-\frac{1}{4}(x-3)=0$,
$x-\frac{1}{2}(y-1)-\frac{1}{3}(x-2)=0$.

18. $\frac{x-2}{3}-\frac{y+2}{4}=0$,
$\frac{2x-5}{5}-\frac{11-2y}{7}=0$.

19. $\frac{x-2}{3}-\frac{y+5}{2}=0$,
$\frac{2x-7}{3}-\frac{13-y}{16}=0$.

88. Instead of proceeding as in Art. 86 we may solve two simultaneous equations in the following manner.

Example. Solve the equations $3x-5y=2$ and $5x-2y=16$.

From the first equation we have $3x=5y+2$, and therefore $x=\frac{1}{3}(5y+2)$. Now substitute this value of x in the second equation, and we get

$$\tfrac{5}{3}(5y+2)-2y=16,$$

which is a simple equation containing only *one* of the unknown quantities; the solution of which gives $y=2$. Then, since $y=2$, we have

$$x=\tfrac{1}{3}(5y+2)=\tfrac{1}{3}(10+2)=4.$$

We may also proceed as follows. From the first of the given equations we have $3x=5y+2$, or $x=\frac{1}{3}(5y+2)$. Also from the second equation we have $5x=2y+16$, or $x=\frac{1}{5}(2y+16)$.

Hence, by equating the two values of x, we get

$$\tfrac{1}{3}(5y+2)=\tfrac{1}{5}(2y+16).$$

The last equation gives $y=2$, and then as before we find that $x=4$.

89. The following are examples of common types of simultaneous equations:

Ex. 1. Solve $57x+25y=3772$ and $25x+57y=1148$.

We have
$$57x+25y=3772,$$
$$25x+57y=1148,$$

In this example the direct application of the method of Art. 86 would lead to troublesome arithmetic; but, since the coefficients of x and y are simply interchanged in the two given equations, we should by addition, or subtraction, obtain equations in which the coefficients of x and y would be equal. Hence we proceed thus:

By addition, $$82x+82y=4920;$$
$$\therefore\ x+y=4920\div 82=60.$$

By subtraction, $$32x-32y=2624;$$
$$\therefore\ x-y=2624\div 32=82.$$

And from the equations $x+y=60$, $x-y=82$ we obtain at once $x=71$ and $y=-11$.

Ex. 2. Solve $(x-1)(y-2)-(x-2)(y-1)=-2$,
$$(x+2)(y+2)-(x-2)(y-2)=32.$$

On removing the brackets, the first equation becomes

$$xy-y-2x+2-xy+2y+x-2=-2,$$

or
$$-x+y=-2 \text{ .... (i).}$$

The second equation becomes

$$xy+2x+2y+4-xy+2x+2y-4=32,$$

or
$$4x+4y=32 \text{ .... (ii).}$$

Multiplying (i) by 4, we have

$$-4x+4y=-8 \text{ .... (iii).}$$

From (ii) and (iii) we have by addition

$$8y=24, \text{ and therefore } y=3.$$

From (ii) and (iii) we have by subtraction

$$8x=40, \text{ and therefore } x=5.$$

Ex. 3. Solve $\frac{3}{x}+\frac{4}{y}=8, \quad \frac{5}{x}+\frac{6}{y}=13.$

These equations can be solved as two simultaneous equations containing the unknown quantities $\frac{1}{x}$ and $\frac{1}{y}$.

Multiply the equations by 5 and 3 respectively; then we have

$$\frac{15}{x}+\frac{20}{y}=40,$$

and

$$\frac{15}{x}+\frac{18}{y}=39.$$

Hence we have by subtraction

$$\frac{2}{y}=1; \text{ and therefore } y=2.$$

We then have from the first equation

$$\frac{3}{x}+\frac{4}{2}=8,$$

that is

$$\frac{3}{x}=8-2=6;$$

$$\therefore \quad 3=6x, \text{ or } x=\tfrac{1}{2}.$$

Thus $x=\frac{1}{2}$, $y=2$ are the values required.

Ex. 4. Solve $ax+by=2ab, \quad bx-ay=b^2-a^2.$

Multiply the first equation by a, and the second equation by b; we then have

$$a^2x+aby=2a^2b,$$

and

$$b^2x-aby=b^3-a^2b.$$

Hence by addition $a^2x+b^2x=a^2b+b^3$,

that is

$$(a^2+b^2)\,x=b\,(a^2+b^2);$$

$$\therefore \quad x=b.$$

Put this value for x in the first of the given equations; then

$$ab+by=2ab;$$

$$\therefore \quad by=ab;$$

$$\therefore \quad y=a.$$

Thus $x=b$, $y=a$ are the required values.

Ex. 5. Solve $5x+3y-13=3x-2y+7=2x+6y-4$.

Since $5x+3y-13=3x-2y+7$,

we have $2x+5y=20$ ....(i).

Also, since $3x-2y+7=2x+6y-4$,

we have $x-8y=-11$ ....(ii).

The solution of (i) and (ii) by the method of Art. 86, gives $x=5$ and $y=2$.

EXAMPLES. XIX.

Solve the equations:

1. $17x+23y=86$, $23x+17y=74$.

2. $15x+19y=18$, $19x+15y=50$.

3. $51x-14y=287$, $14x-51y=-157$.

4. $29x+85y=31$, $13x-43y=95$.

5. $x+{\cdot}2y={\cdot}3$, $1{\cdot}7x+{\cdot}01y={\cdot}345$.

6. ${\cdot}5x+y=2{\cdot}75$, $3{\cdot}4x+{\cdot}02y=1{\cdot}75$.

7. $3x-4y+2=5x-6y-2=7x+2y+4$.

8. $4x-6y-3=7x+2y-4=-2x+3y+24$.

9. $3x+\frac{7}{2}y-2=11y-\frac{2x}{5}=20$.

10. $5x+\frac{y}{5}-1=3y+\frac{x}{3}-2=4$.

11. $\frac{7+x}{3}=\frac{9+y}{5}=\frac{11+x+y}{7}$.

12. $\frac{x+1}{2}=\frac{y+2}{3}=\frac{x+y}{4}$.

13. $(x+1)(y+5)=(x+5)(y+1)$, $xy+x+y=(x+2)(y+2)$.

14. $xy-(x-1)(y-1)=6(y-1)$, $x-y=1$.

15. $\frac{9}{x}-\frac{4}{y}=2$, $\frac{18}{x}+\frac{8}{y}=10$.

16. $\frac{2}{x}+\frac{5}{y}=7$, $\frac{3}{x}-\frac{2}{y}=11$.

17. $\frac{3}{x}+\frac{4}{y}=3$, $\frac{6}{x}-\frac{2}{y}=1$.

18. $\frac{6}{x}-\frac{5}{y}=9$, $\frac{7}{x}-\frac{2}{y}=5$.

19. $x+\frac{3}{y}=\frac{7}{2}$, $3x-\frac{2}{y}=\frac{26}{3}$.

20. $2x-\frac{3}{y}=3$, $8x+\frac{15}{y}+6=0$.

21. $\frac{3}{x}-3y=8$, $\frac{7}{x}+y=5$.

22. $\frac{7}{x}-9y=57$, $\frac{5}{x}+2y=7$.

23. $x-\frac{5}{y}+7=3x-\frac{9}{y}-11=7x+21.$

24. $x+y=2a,$
$x-y=2b.$

25. $ax+by=(a+b)^2,$
$ax-by=a^2-b^2.$

26. $ax+by=a^2+b^2,$
$bx+ay=2ab.$

27. $x-\ \ y=\ a\ -\ b,$
$ax-by=2a^2-2b^2.$

28. $ax+by=a^2-b^2,$
$bx+ay=a^2-b^2.$

29. $b^2x-a^2y=0,$
$bx\ +ay\ =a+b.$

30. $x+\ \ y=a+b,$
$ax-by=b^2-a^2.$

31. $bx-ay=b^3,$
$ax-by=a^3.$

32. $ax-by=a^2+b^2,$
$x+\ \ y=2a.$

33. $a^2x+b^2y=c^2,$
$a^3x+b^3y=c^3.$

34. $ax+by=1,$
$bx+ay=1.$

35. $\frac{a}{x}+\frac{b}{y}=1,$
$\frac{b}{x}+\frac{a}{y}=1.$

36. $(a+b)\,x+(a+c)\,y=a+b,$
$(a+c)\,x+(a+b)\,y=a+c.$

37. $(a+b)\,x-(a-b)\,y=3ab,$
$(a+b)\,y-(a-b)\,x=ab.$

90. **Simultaneous Equations with three unknown quantities.** Three simultaneous equations of the first degree containing the three unknown quantities x, y and z can be solved in the following manner.

Multiply the first and second of the given equations by such quantities that in the resulting equations the coefficients of one of the unknown quantities, z suppose, may be equal: then by addition or subtraction we eliminate z.

Then take the first and third, or the second and third of the given equations, and eliminate z in a similar manner. We thus obtain *two* Simultaneous equations containing the *two* unknown quantities x and y; and these can be solved as in Art. 86.

Ex. 1. Solve the equations

$$2x+4y+z=7,$$
$$3x+2y+2z=8,$$
$$5x-4y+4z=9.$$

Multiply the first equation by 2; then

$$4x+8y+2z=14.$$

Subtract the second equation; then

$$x+6y=6 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Now multiply the first equation by 4; then

$$8x+16y+4z=28.$$

Subtract the third equation; then

$$3x+20y=19 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

From (i) and (ii) we obtain $x=3$, $y=\frac{1}{2}$. Substitute these values in the first of the given equations; then

$$6+2+z=7; \text{ therefore } z=-1.$$

Thus $x=3$, $y=\frac{1}{2}$ and $z=-1$ are the values required.

Ex. 2. Solve the equations

$$\frac{1}{y}+\frac{1}{z}-\frac{1}{x}=5,$$
$$\frac{1}{z}+\frac{1}{x}-\frac{1}{y}=3,$$
$$\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=1.$$

Add the first two equations; then

$$\frac{2}{z}=8, \text{ whence } z=\frac{1}{4}.$$

Add the first and third equations; then

$$\frac{2}{y}=6, \text{ whence } y=\frac{1}{3}.$$

Add the second and third equations; then

$$\frac{2}{x}=4, \text{ whence } x=\frac{1}{2}.$$

EXAMPLES. XX.

1. $$\begin{aligned} y+z&=14,\\ z+x&=18,\\ x+y&=24. \end{aligned}$$

2. $$\begin{aligned} y+z&=2a,\\ z+x&=2b,\\ x+y&=2c. \end{aligned}$$

3. $$\begin{aligned} x+y+z&=1,\\ 2x+3y+z&=4,\\ 4x+9y+z&=16. \end{aligned}$$

4. $$\begin{aligned} 5x+3y+7z&=2,\\ 2x-4y+9z&=7,\\ 3x+2y+6z&=3. \end{aligned}$$

5. $$\begin{aligned} 2x-y+z&=4,\\ 5x+y+3z&=5,\\ 2x-3y+4z&=20. \end{aligned}$$

6. $$\begin{aligned} x+2y-3z&=6,\\ 2x+4y-7z&=9,\\ 3x-y-5z&=8. \end{aligned}$$

7. $$\begin{aligned} x+2y+3z&=4,\\ 2x+3y+4z&=6,\\ 3x+4y+5z&=8. \end{aligned}$$

8. $$\begin{aligned} 3x+2y+5z&=21,\\ 2x-3y+4z&=11,\\ x+3y+7z&=20, \end{aligned}$$

9. $$\begin{aligned} 10x-2y+4z&=5,\\ 3x+5y-3z&=7,\\ x+3y-2z&=2. \end{aligned}$$

10. $$\begin{aligned} x+y+z&=1,\\ \frac{x}{2}+\frac{y}{4}+4z&=1,\\ \frac{5x}{3}+\frac{3y}{4}-\frac{z}{2}&=0. \end{aligned}$$

11. $$\begin{aligned} ax+by&=1,\\ by+cz&=1,\\ cz+ax&=1. \end{aligned}$$

12. $$\begin{aligned} cy+bz&=bc,\\ az+cx&=ca,\\ bx+ay&=ab. \end{aligned}$$

CHAPTER VIII.

PROBLEMS.

91. WE shall now give examples of problems which involve more than one unknown quantity, and in which the relations between the known and unknown quantities are expressed algebraically by means of equations of the first degree.

Many of the problems given in Chapter VI. really contain *two* unknown quantities, but the given relations are in those cases of so simple a nature that it is easy to find an equation giving *one* of the unknown quantities in terms of the known quantities, and having found one of the unknown quantities the other is immediately determined.

Ex. 1. Find two numbers such that the greater exceeds twice the less by three, and that twice the greater exceeds the less by 27.

Let x and y be the numbers, of which x is the greater.

Then we have by the question

$$x - 2y = 3,$$

and

$$2x - y = 27.$$

Multiply the first equation by 2; then

$$2x - 4y = 6.$$

Now subtract the members of this last equation from the corresponding members of the second equation, and we have

$$3y = 21, \text{ or } y = 7.$$

Then from the first equation

$$x=3+2y=3+14=17.$$

Thus the numbers are 17 and 7.

Ex. 2. A number of two digits is equal to seven times the sum of its digits, and the digit in the ten's place is greater by four than the digit in the unit's place. What is the number?

Let x be the digit in the ten's place, and y be the digit in the unit's place.

Then the number is equal to $10x+y$, for the x represents so many *tens*; also the sum of the digits is $x+y$.

Hence we have $10x+y=7(x+y)$,

that is $10x+y=7x+7y$;

$\therefore$ $3x=6y$, or $x=2y$.

We have also $x=y+4$.

Hence $2y=y+4$, or $y=4$,

and therefore $x=8$.

Thus the required number is 84.

Ex. 3. Find the fraction which is equal to $\frac{1}{2}$ when its numerator is increased by unity, and is equal to $\frac{1}{3}$ when its denominator is increased by unity.

Let $x=$ the numerator of the fraction, and $y=$ the denominator.

Then we have by the question

$$\frac{x+1}{y}=\frac{1}{2}, \text{ and } \frac{x}{y+1}=\frac{1}{3}.$$

Multiply the first equation by y; then

$$x+1=\frac{y}{2}.$$

Multiply the second equation by $y+1$; then

$$x=\frac{1}{3}(y+1).$$

Subtract the corresponding members of the last two equations; and we have

$$1=\frac{y}{2}-\frac{1}{3}(y+1),$$

from which we find that $y=8$. Then, since y is 8,

$$x=\frac{y}{2}-1=4-1=3.$$

Thus the fraction is $\frac{3}{8}$.

Ex. 4. A man and a boy can do in 15 days a piece of work which would be done in 2 days by 7 men and 9 boys. How long would it take one man to do it?

Let $x=$ the number of days in which one man would do the whole;

and let $y=$ the number of days in which one boy would do the whole.

Then a man does $\frac{1}{x}$th of the whole in a day;

and a boy does $\frac{1}{y}$th of the whole in a day.

Now by the question a man and a boy together do $\frac{1}{15}$th of the whole in a day.

Hence we have $$\frac{1}{x}+\frac{1}{y}=\frac{1}{15}.$$

We have also, since 7 men and 9 boys do half the work in a day

$$\frac{7}{x}+\frac{9}{y}=\frac{1}{2}.$$

Multiply the first equation by 9 and subtract the second; then

$$\frac{9}{x}-\frac{7}{x}=\frac{9}{15}-\frac{1}{2}$$

that is $$\frac{2}{x}=\frac{1}{10}, \text{ or } \frac{1}{x}=\frac{1}{20}.$$

$$\therefore \qquad x=20.$$

Thus one man would do the work in 20 days.

EXAMPLES. XXI.

1. *A* and *B* have £50 between them, but if *A* were to lose half his money, and *B* $\frac{2}{3}$ of his, they would then have only £20. How much has each?

2. A number of two digits has its digits reversed by the addition of 9. Shew that the digits differ by unity.

3. A man bought 8 cows and 50 sheep for £225. He sold the cows at a profit of 20 per cent., and the sheep at a profit of 10 per cent., and received in all £257. 10*s*. What was the cost of each cow and of each sheep?

4. Twenty-eight tons of goods are to be carried in carts and waggons, and it is found that this will require 15 carts and 12 waggons, or else 24 carts and 8 waggons. How much can each cart and each waggon carry?

5. *A* and *B* can perform a certain task in 30 days, working together. After 12 days however *B* is called off, and *A* finished it by himself 24 days after. How long would each take to do the work alone?

6. If the numerator of a certain fraction be increased by 1 and its denominator diminished by 1, its value will be 1. If the numerator be increased by the denominator and the denominator be diminished by the numerator, its value will be 4. Find the fraction.

7. Find the fraction such that if you quadruple the numerator and add 3 to the denominator the fraction is doubled, but if you add 2 to the numerator and quadruple the denominator the fraction is halved.

8. The first edition of a book had 600 pages, and was divided into two parts. In the second edition one quarter of the second part was omitted and 30 pages added to the first. The change made the two parts of the same length. What were they in the first edition?

9. If *A* were to receive £10 from *B* he would then have twice as much as *B* would have left; but if *B* were to receive £10 from *A*, *B* would have three times as much as *A* would have left. How much has each?

10. A farmer sold 30 bushels of wheat and 50 bushels of barley for £18. 15*s*. He also sold at the same prices 50 bushels of wheat and 30 bushels of barley for £19. 5*s*. What was the price of the wheat per bushel?

11. A rectangle is of the same area as another which is 6 yards longer and 4 yards narrower; it is also of the same area as a third, which is 8 yards longer and 5 yards narrower. What is its area?

12. *A* and *B* can together do a piece of work in 15 days. After working together for 6 days, *A* went away, and *B* finished it by himself 24 days after. In what time would *A* alone do the whole?

13. An income of £120 a year is derived from a sum of money invested partly in $3\frac{1}{2}$ per cent. stock and partly in 4 per cent. stock. If the stock be sold out when the $3\frac{1}{2}$ per cents. are at 108 and the 4 per cents. at 120, the capital realised is £3672. How much stock of each kind was there?

14. A number of two digits is equal to seven times the sum of its digits: shew that one digit must be twice the other.

15. Find all the numbers of two digits, each of which is equal to four times the sum of the digits.

16. £1000 is divided between *A*, *B*, *C* and *D*. *B* gets half as much as *A*; the excess of *C*'s share over *D*'s share is equal to one-third of *A*'s share, and if *B*'s share were increased by £100 he would have as much as *C* and *D* have between them. Find how much each gets.

17. A number has two digits of which the second is double of the first; and, if the digits be reversed, the new number exceeds the original number by 36; find the number.

18. A certain number consists of two digits, and another number is formed from it by reading it backwards. If the sum of the two numbers is 99 and the difference is 45, find the digits.

19. In a certain proper fraction the difference between the numerator and the denominator is 12, and if each be increased by 5 the fraction becomes equal to $\frac{3}{4}$. Find it.

20. The wages of 10 men and 8 boys for a day amount to £1. 17*s*.; and four men receive one shilling more than six boys. How much does each boy receive?

21. A farmer has two farms for each of which he pays a rent of 30 shillings an acre, and his total rent is £675. If the rent of one farm were reduced by 5 shillings an acre, and that of the other by 10 shillings an acre, his rent would be £500. What is the acreage of each of the farms?

22. A man has one pound's worth of silver in half-crowns, shillings and sixpences; and he has in all 20 coins. If he changed the sixpences for pennies, and the shillings for sixpences, he would have 73 coins. How many coins of each kind has he?

23. The price of a passenger's ticket on a French railway is proportional to the distance he travels; he is allowed 25 kilogrammes of luggage free, but on every kilogramme beyond this amount he is charged a sum proportional to the distance he goes. If a journey of 200 miles with 50 kilos. of luggage cost 25 francs, and a journey of 150 miles with 35 kilos. cost $16\frac{1}{2}$ francs, what will a journey of 100 miles with 100 kilos. cost?

24. *A* has twice as many pennies as shillings; *B*, who has 8*d.* more than *A*, has twice as many shillings as pennies; together they have one more penny than they have shillings. How much has each?

25. Of the candidates in a certain examination one quarter fail. The number of marks required for passing is less by 2 than the average marks obtained by all the candidates, less by 11 than the average marks of those who pass, and equal to double the average marks of those who fail. How many marks are required for passing?

MISCELLANEOUS EXAMPLES. II.

A. **1.** Find the value of

$$\frac{ab+2bc-3cd}{b+c+d}+\frac{a^3+b^3-c^3}{a^2+b^2-c^2}, \text{ when } a=1,\ b=2,\ c=-3 \text{ and } d=0.$$

2. Subtract $2a-3(a-\overline{b-a})$ from $2b-3(b-\overline{a-b})$.

3. Shew that

$$(x+3)(y+3)-3(x+1)(y+1)+3(x-1)(y-1)-(x-3)(y-3)=0,$$

and that

$$(x+2)(y+2)-4(x+1)(y+1)+6xy-4(x-1)(y-1)+(x-2)(y-2)=0.$$

4. Divide $x^5-5x^4+7x^3-x^2-4x+2$ by x^3-3x^2+3x-1.

5. Solve the equations:

(i) $\frac{x}{2}+\frac{x+1}{3}+\frac{x+2}{4}+\frac{x+3}{5}=4.$

(ii) $\frac{x}{3}+\frac{y}{5}=5,\ 2x+\frac{y}{3}-17=0.$

(iii) $\left.\begin{aligned} ax-by&=a^2-b^2 \\ bx-ay&=b^2-a^2 \end{aligned}\right\}.$

6. A number of marbles was divided among three boys so that the first boy had 10 less than half of the whole, the second had 10 more than a quarter of what was left, and the third had 20. How many marbles were there?

B. 1. Find the numerical value of

$$(a-b)^2+(b-c)^2+(c-a)^2+(b-c)(c-a)(a-b),$$

when $a=1$, $b=2$ and $c=\frac{1}{2}$.

2. From the sum of $(2a-b)^2$ and $(a-2b)^2$ take the square of $2(a-b)$.

3. Shew that

$$n^3=n(n-1)(n-2)+3n(n-1)+n,$$

and $$n^4=n(n-1)(n-2)(n-3)+6n(n-1)(n-2)+7n(n-1)+n.$$

4. Find the algebraical expression which when divided by x^2-2x+1 gives a quotient x^2+2x+1 and a remainder $x+1$.

5. Solve

(i) $3(x+3)^2+5(x+5)^2=8(x+8)^2.$

(ii) $\left.\begin{aligned} \tfrac{1}{2}(x+y)-\tfrac{1}{3}(x-y)&=8 \\ \tfrac{1}{3}(x+y)-\tfrac{1}{4}(x-y)&=11 \end{aligned}\right\}.$

(iii) $\frac{1}{x}+\frac{4}{y}=1,\quad \frac{7}{x}-\frac{8}{y}=\frac{1}{6}.$

6. A man paid a bill of £100 with sovereigns and crowns, using in all 130 coins. How many coins of each sort did he use?

C. 1. Simplify $a - [a + b - \{a + b + c - \overline{a + b + c + d}\}]$.

2. Find the sum, the difference, and the product of $a + b$ and $\frac{a}{2} - \frac{b}{2}$.

3. Shew that

$$ab - 3(a-1)(b-1) + 3(a-2)(b-2) - (a-3)(b-3) = 0,$$

and that

$$ab - 4(a-1)(b-1) + 6(a-2)(b-2) - 4(a-3)(b-3) + (a-4)(b-4) = 0.$$

4. Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$, and from the result write down, without division, the quotient when $8x^3 + 8y^3 + z^3 - 12xyz$ is divided by $2x + 2y + z$.

5. Solve the equations:

(i) $x = \frac{1}{3}(5y + 2)$, $y = \frac{1}{2}(x - 1)$.

(ii) $ax + by = a^3$, $bx + ay = b^3$.

6. A and B each shoot 30 arrows at a target. B makes twice as many hits as A, and A makes three times as many misses as B. Find the number of hits and misses of each. 4

D. 1. Find the value of

$$abc(ab + bc + cd + da) \div (a + b)(a + c)(a + d),$$

when $a = 1$, $b = 3$, $c = -5$ and $d = 0$.

2. Simplify $(a + b)^2 - [2a^2 - \{(a - b)(a + 2b) - b(a - b)\}]$.

3. Find the continued product of $x + a$, $x + b$ and $x + c$; and from the result *write down* the continued product of $a - x$, $a - y$ and $a - z$.

4. Divide $75a^2b^3c^5 - 15a^3b^4c^3$ by $5a^2bc^2$, and

$a^2 - 2b^2 - 6c^2 + ab - ac + 7bc$ by $a - b + 2c$.

5. Solve the equations:

(i) $\frac{1}{2}(x - 1) - \frac{1}{3}(x - 2) + \frac{1}{4}(x - 3) = 0$.

(ii) $\frac{x - 3}{2} = \frac{y - 1}{3} = \frac{x + y}{9}$.

(iii) $\frac{3}{x} + \frac{5}{y} = 22$, $\frac{7}{x} - \frac{4}{y} = 20$.

6. A square grass plot would contain 69 square feet less if each side were one foot shorter. How many square feet does it contain?

E. 1. Simplify

$$b^2+\{b(a-c)+ac\}-a\{b-\overline{a-c}\}+b(c-\overline{b-a})-ab.$$

2. From what must the sum of $3a^2+2ab-ac$, $3b^2+2bc-ab$ and $3c^2+2ca-bc$ be taken in order that the remainder may be $a^2+b^2+c^2$?

3. Shew that

$$(x+y)^2+(y+z)^2+(z+x)^2$$
$$+2(x+y)(x+z)+2(y+z)(y+x)+2(z+x)(z+y)$$
$$=4(x+y+z)^2.$$

4. Divide $\frac{1}{3}x^3+\frac{11}{6}x^2-\frac{7}{9}x+\frac{4}{3}$ by $\frac{1}{3}x+2$, and

$$acx^3+(ad-bc)x^2-(ac+bd)x+bc \text{ by } ax-b.$$

5. Solve the equations:

(i) $(x-1)(x+2)=x^2+3.$

(ii) $\left.\begin{aligned} 2x+\cdot 4y&=1\cdot 2\\ 5x+\cdot 2y&=1\cdot 8\end{aligned}\right\}.$

(iii) $\left.\begin{aligned} (a-b)x&=(a+b)y\\ x+y&=2a\end{aligned}\right\}.$

6. Find two numbers such that twice their difference is greater by unity than the smaller number, and is less by two than the larger number.

F. 1. Find the value of $3(x+y+z)^3-(x^3+y^3+z^3)$, when $x=3$, $y=-5$ and $z=7$.

2. Find the continued product of a^4+x^4, a^2+x^2, $a+x$ and $a-x$.

3. Shew that $(1-a)(1-b)+a(1-b)+b=1$,

and that $(1-a)(1-b)(1-c)+a(1-b)(1-c)+b(1-c)+c=1.$

$= 1-c$

4. Divide $4y^4-9y^2+6y-1$ by $2y^2+3y-1$.

Divide also $a^2+2b^2-3c^2+bc+2ac+3ab$ by $a+b-c$.

5. Solve the equations:

(i) $(a+x)(b+x)=(c+x)(d+x).$

(ii) $3x-7y=4,\ \ 7x-9y=2.$

6. If 12 lbs. of tea and 3 lbs. of coffee cost £1. 15*s*., and 12 lbs. of coffee and 3 lbs. of tea cost £1. 7*s*. 6*d*., what is the price per lb. of tea and of coffee?

CHAPTER IX.

FACTORS.

92. **Definitions.** An algebraical expression which does not contain any *letter* in the denominator of any term is called an **integral expression.**

Thus $a+b$ and $\frac{1}{2}a^2-\frac{1}{3}b^2$ are integral expressions.

An expression is said to be **integral with respect to any particular letter** when that letter does not occur in the denominator of any term.

Thus $\frac{x^2}{a}+\frac{x}{a+b}$ is *integral with respect to* x.

An expression is said to be **rational** when none of its terms contain square or other roots.

93. In the present chapter we shall shew how factors of rational and integral algebraical expressions can be found in certain simple cases.

In arithmetic we mean by the factors of a number its integral divisors only; and similarly, by the *factors* of an algebraical expression, we mean the rational and integral expressions which exactly divide it.

94. **Monomial Factors.** When any letter, or group of letters, is common to all the terms of an expression, each term, and therefore the whole expression is divisible by that letter, or group of letters.

Thus $ab+ac=a(b+c)$,

$a^2b+ab^2=ab(a+b)$,

and $3ax^3y+6ax^2y^3=3ax^2y(x+2y^2)$.

Monomial factors, if there are any, can be seen by inspection, and the whole expression can be at once written as the product of the monomial factor and its co-factor, as in the above examples. In what follows therefore we need only consider expressions which have been freed from monomial factors.

EXAMPLES. XXII.

Find the factors of the following expressions:

1. x^2+x. 2. a^2-ab. 3. $ab-bc$.
4. $2ax+\frac{1}{2}x^2$. 5. $4x^3-3x^2$. 6. a^3-3a^2b.
7. $x^4-5x^3y+20x^2y^2$. 8. $a^6-a^4x+a^2x^2$.
9. $6ax^4-5a^3x^2+20a^2x^3$. 10. $3a^3x^4y^2-\frac{1}{9}a^2x^3y^3$.
11. $11a^2bc^2-\frac{1}{5}ab^2c^3$. 12. $p^2q^3r^6-7p^6q^5$.

95. **Factors found by comparing with known identities.** Sometimes an algebraical expression is of the *same form* as some known result of multiplication: in this case its factors can be written down. We proceed to apply this principle in the case of the most important forms of algebraical expressions.

96. We know that

$$a^2+2ab+b^2=(a+b)^2,$$

and

$$a^2-2ab+b^2=(a-b)^2.$$

Hence, when a trinomial expression consists of the *sum of the squares of any two quantities plus twice the product of the quantities*, it is equal to the *square of their sum*. And when a trinomial expression consists of the *sum of the squares of any two quantities minus twice their product*, it is equal to the *square of their difference*.

Hence it is easy to recognise when a trinomial is a complete square; for two of the terms must be squares, and the remaining term must be twice the product of the quantities whose squares are the other terms.

The two terms which are squares must both have the same sign.

Thus $a^2+6ab+9b^2$, that is $a^2+(3b)^2+2a\,(3b)$, consists of the squares of a and $3b$ together with twice the product of a and $3b$, and hence $$a^2+6ab+9b^2=(a+3b)^2.$$

Again
$$4ab-a^2-4b^2=-(a^2+4b^2-4ab)=-\{a^2+(2b)^2-2a\,(2b)\}=-(a-2b)^2.$$

As other examples, we have
$$16a^4+8a^2+1=(4a^2)^2+2\,(4a^2)+1=(4a^2+1)^2,$$
$$x^3-4x^2y^2+4xy^4=x\,(x^2-4xy^2+4y^4)\}=x\,\{x^2-2x\,(2y^2)+(2y^2)^2\}$$
$$=x\,(x-2y^2)^2,$$
and $$(a+b)^2-2c\,(a+b)+c^2=\{(a+b)-c\}^2.$$

Note. We may consider that $a^2-2ab+b^2$ is equal to $(b-a)^2$ instead of $(a-b)^2$, for by the Law of Signs $(b-a)^2$, that is $\{-(a-b)\}^2$, is equal to $(a-b)^2$. In fact when we find that an expression is equal to the product of two factors, we may equally well consider it as the product of the same two factors with all the signs changed.

EXAMPLES. XXIII.

Find the factors of the following expressions :—

1. $4x^2+4x+1$.
2. $9x^2-6x+1$.
3. $1-8x^2+16x^4$.
4. $4a^2-12ab+9b^2$.
5. $9a^4+24a^2b^2+16b^4$.
6. $x^2+xy+\frac{1}{4}y^2$.
7. $4a^2x^2+4abxy+b^2y^2$.
8. $25a^4x^2-30a^2b^2xy+9b^4y^2$.
9. $3a^2+6ab+3b^2$.
10. $5a^4-10a^2b+5b^2$.
11. $a^3-6a^2b+9ab^2$.
12. $3a^5-30a^4b^3+75a^3b^6$.
13. $4x^2y^2-x^4-4y^4$.
14. $8x^2-4x^4-4$.
15. $4xy^3-4x^2y^2+x^3y$.
16. $x^2y^2+x^3y+\frac{1}{4}xy^3$.
17. $(a+b)^2+4c\,(a+b)+4c^2$.
18. $(x^2+y^2)^2-2\,(x^2+y^2)\,z^2+z^4$.
19. $4x^2y^2+4\,(a+b)\,xy+(a+b)^2$.
20. $9\,(a+b)^2-6c\,(a+b)+c^2$.

97. From the formula
$$a^2-b^2=(a+b)\,(a-b),$$
we see that the *difference of the squares of any two quantities is equal to the product of the sum and the difference of the quantities.*

Thus a^2-4b^2, or $a^2-(2b)^2$, is equal to $(a+2b)(a-2b)$.

Also $9x^3-4xy^2$, that is $x\{(3x)^2-(2y)^2\}$, is equal to

$$x(3x+2y)(3x-2y).$$

As other examples, we have:

$$8axy-18a^3x^3y^3=2axy(4-9a^2x^2y^2)=2axy\{2^2-(3axy)^2\}$$
$$=2axy(2-3axy)(2+3axy),$$
$$a^4-b^4=(a^2+b^2)(a^2-b^2)=(a^2+b^2)(a+b)(a-b),$$

and $$79^2-71^2=(79+71)(79-71)=150\times 8=1200.$$

We may deal with the squares of multinomial expressions in precisely the same way as with the squares of monomial expressions.

Thus $$(a+b)^2-c^2=\{(a+b)+c\}\{(a+b)-c\}$$
$$=(a+b+c)(a+b-c),$$
$$a^4+a^2b^2+b^4=(a^2+b^2)^2-(ab)^2=(a^2+b^2+ab)(a^2+b^2-ab),$$

and $$(2a-b+2c)^2-(a+4b+c)^2$$
$$=\{(2a-b+2c)+(a+4b+c)\}\{(2a-b+2c)-(a+4b+c)\}$$
$$=(3a+3b+3c)(a-5b+c).$$

EXAMPLES. XXIV.

Find the factors of the following expressions:—

1. a^2-9.
2. $16-b^2$.
3. $25a^2-b^2$.
4. x^2-9y^2.
5. $16x^2-9y^2$.
6. $64a^2-49b^2$.
7. $4a^2-81b^2$.
8. x^2-9y^4.
9. $36a^4-49b^2$.
10. $4a^2b^2-9c^2$.
11. $9a^2x^2-49b^2y^2$.
12. $49a^2b^2c^2-36x^2y^2z^2$.
13. $4xy^2-9x^3$.
14. $8ab^2-18a^3$.
15. $3a^5-108a^3$.
16. $7a^5-28a^9$.
17. $8x^3y-32xy^3$.
18. $7abc^2-7a^3b^3$.
19. x^4-y^4.
20. x^4-16y^4.
21. $81a^4-16b^4$.
22. $625a^4-256x^4$.
23. $x^4y^4-a^4b^4$.
24. $a^4b^4-81c^4d^4$.
25. $81x^4y^4-1$.
26. $16a^4b^4c^4-1$.
27. $16-81x^4y^4$.
28. x^8-y^8.
29. $a^8-b^8c^8$.
30. $a^{10}-a^2$.
31. $(a+b)^2-c^2$.
32. $(a+b)^2-4c^2$.
33. $4(x+y)^2-1$.
34. $9(x-y)^2-4$.
35. $(x+y)^2-(x-y)^2$.
36. $(2a+b)^2-(2b+a)^2$.
37. $x^2-(x-y)^2$.
38. $a^2-(2b-a)^2$.
39. $4(a+b)^2-(a-b)^2$.
40. $9(x+y)^2-4(x-y)^2$.
41. $(a^2+b^2)^2-4a^2b^2$.
42. $(a+b+c)^2-(a-b-c)^2$.
43. $(3a+b-2c)^2-(a+3b-c)^2$.
44. $(a-2b+3c)^2-(a-c)^2$.
45. $(3x^2+x-2)^2-(x^2-x-2)^2$

98. From the formulae

$$x^3 + a^3 = (x + a)\,(x^2 - ax + a^2),$$

and $$x^3 - a^3 = (x - a)\,(x^2 + ax + a^2), \qquad \text{[Art. 69.]}$$

we see that the *sum of the cubes* of any two quantities is divisible by the *sum* of the quantities; and that the *difference of the cubes* of any two quantities is divisible by the *difference of the quantities*.

Thus $8a^3 + 27b^3$, that is $(2a)^3 + (3b)^3$, is divisible by $2a + 3b$, and the quotient is $(2a)^2 - (2a)(3b) + (3b)^2$, that is $4a^2 - 6ab + 9b^2$. Also $27x^3 - 8$, that is $(3x)^3 - 2^3$, is equal to

$$(3x - 2)\{(3x)^2 + (2\,.\,3x) + 2^2\} = (3x - 2)(9x^2 + 6x + 4).$$

As other examples,

$$a^3b^3 - \tfrac{1}{8}c^3 = (ab - \tfrac{1}{2}c)(a^2b^2 + \tfrac{1}{2}abc + \tfrac{1}{4}c^2),$$

and
$$\begin{aligned} a^6 - b^6 &= (a^3 + b^3)(a^3 - b^3) \\ &= (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2). \end{aligned}$$

The cubes of multinomial expressions may be dealt with in precisely the same way.

Thus $(a + b)^3 - c^3 = \{(a + b) - c\}\{(a + b)^2 + (a + b)\,c + c^2\}$,

and $(x - 2y)^3 - (y - 2x)^3$
$$\begin{aligned} &= (\overline{x - 2y} - \overline{y - 2x})\,\{(x - 2y)^2 + (x - 2y)(y - 2x) + (y - 2x)^2\} \\ &= (3x - 3y)(3x^2 - 3xy + 3y^2). \end{aligned}$$

EXAMPLES. XXV.

Find factors of each of the following expressions:

1. $a^3 - 8b^3$. **2.** $8a^3 + b^3$. **3.** $8a^3 - 125x^3$.

4. $a^3 - 125x^6$. **5.** $4a^3 + 32b^3$. **6.** $27x^3 - \frac{1}{8}y^3$.

7. $x^3y^3 + \frac{1}{27}a^3b^3$. **8.** $8a^6b^6 + x^6$. **9.** $2xy^3 - \frac{1}{4}x^4$.

10. $9a^4b^2 - \frac{1}{3}ab^5$. **11.** $3a^4b + 24ab^4$. **12.** $40a^3bc - 5b^4c^4$.

13. $a^6 - 64$. **14.** $64a^6 - 729b^6$. **15.** $x^{12} - a^6b^6$.

16. $(x + 2y)^3 - y^3$. **17.** $(x + 2y)^3 + (y + 2x)^3$.

18. $(2y - x)^3 - (2x - y)^3$. **19.** $(x - 3y)^3 - (y - 3x)^3$.

20. $(2y - x)^3 + (2x - y)^3$.

99. By multiplication we have

$$(x + a)(x + b) = x^2 + (a + b)\,x + ab$$

Hence conversely, if an expression of the form x^2+px+q be the product of the two factors $x+a$ and $x+b$, the given expression must be the same as $x^2+(a+b)x+ab$; we must therefore have $p=a+b$ and $q=ab$. Hence *a and b are such that their sum is p, and their product q.*

For example, to find the factors of $x^2+7x+12$. The factors will be $x+a$ and $x+b$, where $ab=12$ and $a+b=7$. Hence we must find two numbers whose product is 12 and whose sum is 7. Pairs of numbers whose product is 12 are 12 and 1, 6 and 2, and 4 and 3; and the sum of the last pair is 7. Hence

$$x^2+7x+12=(x+4)(x+3).$$

Again, to find the factors of $x^2-7x+10$, we have to find two numbers whose product is 10, and whose sum is -7. Since the product is $+10$, the two numbers must *both* be positive or *both* negative; and since the sum is -7, they must both be negative. The pairs of negative numbers whose product is 10 are -10 and -1, and -5 and -2; and the sum of the last pair is -7. Hence

$$x^2-7x+10=(x-5)(x-2).$$

Again, to find the factors of $x^2+3x-18$, we have to find two numbers whose product is -18, and whose sum is 3. Since the product is -18, one of the numbers is positive and the other negative. The pairs of numbers whose product is -18 are -18 and 1, -9 and 2, -6 and 3, -3 and 6, -2 and 9, and -1 and 18; and of these pairs the sum of 6 and -3 is 3. Hence

$$x^2+3x-18=(x+6)(x-3).$$

EXAMPLES. XXVI.

Find the factors of each of the following expressions:

1. x^2+4x+3.
2. x^2-4x+3.
3. x^2-6x+8.
4. $x^2-8x+15$.
5. $x^2-11x+18$.
6. $x^2+9x+20$.
7. x^2+2x-3.
8. x^2+4x-5.
9. x^2+x-6.
10. x^2-x-6.
11. $x^2+2x-35$.
12. $x^2-3x-10$.
13. $x^2+5x-14$.
14. $x^2-x-132$.
15. $x^2+18x+72$.
16. $x^2-5x-84$.
17. $x^2-25x+150$.
18. $x^2+5x-150$.
19. $x^2+11x-180$.
20. $x^2-x-156$.
21. $x^2-31x+240$.
22. $x^2-17x-200$.
23. $x^2-34x+288$.
24. $x^2-35x-200$.

100. By multiplication we have

$$(ax+b)(cx+d)=acx^2+(ad+bc)x+bd.$$

Hence conversely, if an expression of the form px^2+qx+r be the product of the two factors $ax+b$ and $cx+d$, the given expression must be the same as $acx^2+(ad+bc)x+bd$; we must therefore have $ac=p$, $bd=r$ and $ad+bc=q$. We can in simple cases find by trial the values of a, b, c, d which satisfy these relations.

For example, to find the factors of $3x^2-16x+5$. The $3x^2$ can only be given by the multiplication of $3x$ and x. The 5 could only be given by the multiplication of 5 and 1 or -5 and -1; and, since the middle term is negative, the latter pair must be taken. We have now only a choice between $(3x-5)(x-1)$ and $(3x-1)(x-5)$, and it is at once seen that the latter must be taken in order that the coefficient of x in the product may be -16. Thus the factors required are $3x-1$ and $x-5$.

Again, to find the factors of $5x^2+32x-21$, the $5x^2$ can n be given by the multiplication of $5x$ and x. The end term -21 can be the product of -21 and 1, -7 and 3, -3 and 7 or -1 and 21. The possible pairs of factors, so far as the two end terms are concerned, are therefore $(5x\mp21)(x\pm1)$, $(5x\mp7)(x\pm3)$, $(5x\mp3)(x\pm7)$ and $(5x\mp1)(x\pm21)$. It will be found that of these pairs the one which will give $32x$ for the middle term is

$$(5x-3)(x+7).$$

101. The factors of $x^2-5xy+4y^2$ can be found in the same way as the factors of x^2-5x+4. For we must find two quantities whose product is $4y^2$ and whose sum is $-5y$: these are $-4y$ and $-y$. Hence

$$x^2-5xy+4y^2=(x-4y)(x-y).$$

So also the factors of $3x^2-16xy+5y^2$ can be found in the same way as the factors of $3x^2-16x+5$; and the factors of either can be written down as soon as the factors of the other are known.

For example, if we know that $5x^2+32x-21=(5x-3)(x+7)$, we must have $5x^2+32xy-21y^2=(5x-3y)(x+7y)$.

Note. In the above process it is not absolutely necessary to take the *lowest* common multiple of the denominators; *any* common multiple will do equally well, but by using the L.C.M. there is a saving of labour.

133. **Addition of fractions.** The sum (or difference) of two fractions which have the same denominator is a fraction whose numerator is the sum (or difference) of their numerators, and which has the common denominator. This follows from Art. 66.

When two fractions have not the same denominator, they must first be reduced, by the method explained in the last Article, to equivalent fractions which have the same denominator: their sum or difference will then be found by taking the sum or difference of their numerators, retaining the common denominator.

Thus
$$\frac{a}{x}+\frac{b}{x}=\frac{a+b}{x},$$

$$\frac{a}{x}-\frac{b}{x}=\frac{a-b}{x}.$$

$$\frac{a}{x}+\frac{b}{y}=\frac{a\times y}{x\times y}+\frac{b\times x}{y\times x}=\frac{ay}{xy}+\frac{bx}{xy}=\frac{ay+bx}{xy},$$

and
$$a+\frac{b}{y}=\frac{ay}{y}+\frac{b}{y}=\frac{ay+b}{y}.$$

When more than two fractions are to be added, or when there are several fractions some of which are to be added and the others subtracted, the process is precisely the same. The fractions are first reduced to a common denominator, and then the numerators of the reduced fractions are added or subtracted as may be required.

Thus
$$\frac{a}{x}+\frac{b}{x}-\frac{c}{x}=\frac{a+b-c}{x},$$

and $$\frac{a}{x}+\frac{b}{y}-\frac{c}{z}=\frac{a\times yz}{x\times yz}+\frac{b\times xz}{y\times xz}-\frac{c\times xy}{z\times xy}$$

$$=\frac{ayz}{xyz}+\frac{bxz}{xyz}-\frac{cxy}{xyz}=\frac{ayz+bxz-cxy}{xyz}.$$

Note. It may be necessary to remind the student that when there is no sign between a fraction and a letter or number, the sign of *multiplication* is understood. Thus, $2\frac{a}{b}$ means $2\times\frac{a}{b}$ and not $2+\frac{a}{b}$ as in the Arithmetical form $2\frac{1}{2}$.

Ex. 1. Find the value of $\frac{1}{x-a}+\frac{1}{x+a}$.

The L.C.M. of the denominators is $(x-a)(x+a)$; and

$$\frac{1}{x-a}+\frac{1}{x+a}=\frac{x+a}{(x-a)(x+a)}+\frac{x-a}{(x+a)(x-a)}$$

$$=\frac{x+a+(x-a)}{x^2-a^2}=\frac{2x}{x^2-a^2}.$$

Ex. 2. Find the value of $\frac{a}{a-x}+\frac{ax}{x^2-a^2}$.

Beginners should always see that the denominators of the fractions which are to be added are all arranged according to descending powers, or all according to ascending powers, of some particular letter. This is not the case in the present example; but

$$\frac{ax}{x^2-a^2}=\frac{ax}{-(a^2-x^2)}=\frac{-ax}{a^2-x^2}.$$

We then have

$$\frac{a}{a-x}+\frac{-ax}{a^2-x^2}=\frac{a(a+x)}{a^2-x^2}+\frac{-ax}{a^2-x^2}$$

$$=\frac{a(a+x)-ax}{a^2-x^2}$$

$$=\frac{a^2}{a^2-x^2}.$$

Ex. 3. Simplify $\frac{1}{1-x}+\frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}$.

It is sometimes best not to add all the fractions at once; this is particularly the case when the denominators are not all of the same dimensions.

$$\frac{1}{1-x}+\frac{1}{1+x}=\frac{(1+x)+(1-x)}{(1-x)(1+x)}=\frac{2}{1-x^2};$$

then $$\frac{2}{1-x^2}+\frac{2}{1+x^2}=\frac{2(1+x^2)+2(1-x^2)}{(1-x^2)(1+x^2)}=\frac{4}{1-x^4};$$

and finally

$$\frac{4}{1-x^4}+\frac{4}{1+x^4}=\frac{4(1+x^4)+4(1-x^4)}{(1-x^4)(1+x^4)}=\frac{8}{1-x^8};$$

Ex. 4. Find the value of $\frac{1}{x-1}-\frac{1}{x+1}+\frac{1}{x-2}-\frac{1}{x+2}$.

Here again it is best not to reduce all the fractions to a common denominator at once: the work is simplified by proceeding as under.

$$\frac{1}{x-1}-\frac{1}{x+1}=\frac{x+1-(x-1)}{(x-1)(x+1)}=\frac{2}{x^2-1},$$

$$\frac{1}{x-2}-\frac{1}{x+2}=\frac{x+2-(x-2)}{(x-2)(x+2)}=\frac{4}{x^2-4},$$

and $$\frac{2}{x^2-1}+\frac{4}{x^2-4}=\frac{2(x^2-4)+4(x^2-1)}{(x^2-1)(x^2-4)}=\frac{6x^2-12}{x^4-5x^2+4}.$$

Ex. 5. Find the value of $\frac{1}{x^2-5x+6}-\frac{1}{x^2-7x+12}$.

$$\frac{1}{x^2-5x+6}=\frac{1}{(x-2)(x-3)},$$

and $$\frac{1}{x^2-7x+12}=\frac{1}{(x-3)(x-4)};$$

hence the L. C. M. of the denominators is $(x-2)(x-3)(x-4)$.

Hence we have

$$\frac{x-4}{(x-2)(x-3)(x-4)}-\frac{x-2}{(x-2)(x-3)(x-4)}$$

$$=\frac{(x-4)-(x-2)}{(x-2)(x-3)(x-4)}=\frac{-2}{(x-2)(x-3)(x-4)}$$

$$=-\frac{2}{(x-2)(x-3)(x-4)}.$$

Ex. 6. Find the value of

$$\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}.$$

In examples of this kind it is best for beginners to arrange all the factors in the denominators of the fractions so that a precedes b or c, and that b precedes c. We therefore change $b-a$ into $-(a-b)$, $c-a$ into $-(a-c)$, and $c-b$ into $-(b-c)$. Then

$$\frac{1}{(b-a)(b-c)}=\frac{1}{-(a-b)(b-c)}=\frac{-1}{(a-b)(b-c)},$$

and $$\frac{1}{(c-a)(c-b)}=\frac{1}{\{-(a-c)\}\{-(b-c)\}}=\frac{1}{(a-c)(b-c)}.$$

Hence we have to simplify

$$\frac{1}{(a-b)(a-c)}+\frac{-1}{(a-b)(b-c)}+\frac{1}{(a-c)(b-c)}.$$

The L. C. M. is $(a-b)(a-c)(b-c)$; and

$$\frac{1}{(a-b)(a-c)}=\frac{b-c}{(a-b)(a-c)(b-c)},$$

$$\frac{-1}{(a-b)(b-c)}=\frac{-(a-c)}{(a-b)(b-c)(a-c)},$$

and $$\frac{1}{(a-c)(b-c)}=\frac{a-b}{(a-b)(a-c)(b-c)}.$$

Hence we have

$$\frac{b-c-(a-c)+a-b}{(a-b)(a-c)(b-c)}=\frac{b-c-a+c+a-b}{(a-b)(a-c)(b-c)}=\frac{0}{(a-b)(a-c)\,b-c)}=0.$$

EXAMPLES. XXXVIII.

Reduce to their lowest common denominator.

1. $\dfrac{2}{3x}$, $\dfrac{4}{5x}$, $\dfrac{7}{30x}$.
2. $\dfrac{1}{2ax}$, $\dfrac{1}{6bx}$, $\dfrac{1}{8cx}$.
3. $\dfrac{a}{bc}$, $\dfrac{b}{ca}$, $\dfrac{c}{ab}$.
4. $\dfrac{bc}{a}$, $\dfrac{ca}{b}$, $\dfrac{ab}{c}$.
5. $\dfrac{1}{x+1}$, $\dfrac{5}{2x+2}$.
6. $\dfrac{3}{2x-2}$, $\dfrac{4}{3x-3}$.
7. $\dfrac{5}{6x+12}$, $\dfrac{3}{8x+16}$.
8. $\dfrac{1}{x+1}$, $\dfrac{3}{2x+2}$, $\dfrac{5}{x^2-1}$.
9. $\dfrac{6}{5x-5}$, $\dfrac{2}{3x+3}$, $\dfrac{4}{x^2-1}$.
10. $\dfrac{a}{x-a}$, $\dfrac{x}{a-x}$, $\dfrac{a^2}{x^2-a^2}$, $\dfrac{x^2}{a^2-x^2}$.

11. $\frac{2a}{a-b}$, $\frac{b}{2b-2a}$, $\frac{3a^2}{4(a^2-b^2)}$, $\frac{5b^2}{6(b^2-a^2)}$.

12. $\frac{1}{x+1}$, $\frac{2x}{(x+1)^2}$, $\frac{3x^2}{(x+1)^3}$.

13. $\frac{1}{(x-a)(x-b)}$, $\frac{1}{(b-x)(c-x)}$, $\frac{1}{(x-c)(x-a)}$.

14. $\frac{1}{(a-b)(a-c)}$, $\frac{1}{(b-c)(b-a)}$, $\frac{1}{(c-a)(c-b)}$.

Reduce to one term

15. $a-1+\frac{2}{a+1}$.

16. $a+x+\frac{x^2}{a-x}$.

17. $x+2y+\frac{4y^2}{x-2y}$.

18. $\frac{a-b}{a^2b}+\frac{a-b}{ab^2}$.

19. $\frac{a-5b}{5}-\frac{a-3b}{3}$.

20. $\frac{6a-5b}{3}-\frac{4a-7b}{2}$.

21. $\frac{a-3b}{4}+\frac{3a-b}{5}$.

22. $2\frac{x-y}{3}-3\frac{x+y}{4}$.

23. $5\frac{x-2y}{6}-3\frac{y-2x}{4}$.

24. $\frac{x}{4}-\frac{x-4}{3}+\frac{x-5}{6}$.

25. $\frac{2x-3y}{3}+\frac{x+2y}{4}-\frac{3x-2y}{6}$.

Simplify

26. $\frac{a}{a-b}+\frac{b}{b-a}$.

27. $\frac{x}{x-a}+\frac{a}{a-x}$.

28. $\frac{x}{x^2-a^2}+\frac{a}{a^2-x^2}$.

29. $\frac{1}{1-x}-\frac{2}{1-x^2}$.

30. $\frac{1}{2-x}-\frac{4}{4-x^2}$.

31. $\frac{1}{3+x}+\frac{6}{x^2-9}$.

32. $\frac{1}{x-3}-\frac{1}{x-2}$.

33. $\frac{1}{x-4y}-\frac{1}{x-5y}$.

34. $\frac{1}{3x-2y}-\frac{1}{5x-2y}$.

35. $\frac{1+x}{1-x}-\frac{1-x}{1+x}$.

36. $\frac{a+2b}{a-2b}-\frac{a-2b}{a+2b}$.

37. $\frac{1}{x-2}+\frac{2}{(x-2)^2}$.

38. $\frac{a}{a-2b}+\frac{2ab}{(a-2b)^2}$.

39. $\frac{x+y}{x-y}-\frac{x-y}{x+y}+\frac{4y^2}{x^2-y^2}$.

40. $\frac{2a}{a+b}+\frac{2b}{a-b}-\frac{a^2+b^2}{a^2-b^2}$. 41. $\frac{a}{a-1}-1-\frac{1}{a(a-1)}$.

42. $\frac{2}{x-2}-\frac{1}{x+2}-\frac{x+6}{x^2+4}$. 43. $\frac{1}{x-1}-\frac{1}{2(x+1)}-\frac{x+3}{2(x^2+1)}$.

44. $\frac{3-x}{1-3x}-\frac{3+x}{1+3x}-\frac{1-16x}{9x^2-1}$. 45. $\frac{a}{a-x}+\frac{a}{a+x}+\frac{2a^2}{a^2+x^2}$.

46. $\frac{a}{a-x}+\frac{a}{a+x}+\frac{2a^2}{a^2+x^2}+\frac{4a^4}{a^4+x^4}$.

47. $\frac{1}{x+1}-\frac{1}{x+2}+\frac{1}{x+3}-\frac{1}{x+4}$.

48. $\frac{1}{x-1}-\frac{2}{x}+\frac{1}{x+1}$. 49. $\frac{1}{a}-\frac{2}{a+1}+\frac{1}{a+2}$.

50. $\frac{1}{x-3}-\frac{3}{x-1}+\frac{3}{x+1}-\frac{1}{x+3}$. 51. $\frac{1}{a}-\frac{3}{a+1}+\frac{3}{a+2}-\frac{1}{a+3}$.

52. $\frac{1}{x-2}-\frac{4}{x-1}+\frac{6}{x}-\frac{4}{x+1}+\frac{1}{x+2}$.

53. $\frac{1}{a}-\frac{4}{a+1}+\frac{6}{a+2}-\frac{4}{a+3}+\frac{1}{a+4}$.

54. $\frac{1}{x^2-5x+6}-\frac{2}{x^2-4x+3}+\frac{1}{x^2-3x+2}$.

55. $\frac{1}{x^2+5ax+6a^2}-\frac{2}{x^2+4ax+3a^2}+\frac{1}{x^2+3ax+2a^2}$.

56. $\frac{x-1}{(x-2)(x-3)}-\frac{2(x-2)}{(x-3)(x-1)}+\frac{x-3}{(x-1)(x-2)}$.

57. $\frac{1}{x(x-1)}+\frac{2}{1-x^2}+\frac{1}{x(x+1)}$.

58. $\frac{2a}{(x-2a)^2}-\frac{x-a}{x^2-5ax+6a^2}+\frac{2}{x-3a}$.

59. $\frac{3+2a}{2-a}-\frac{2-3a}{2+a}+\frac{16a-a^2}{a^2-4}$. 60. $\frac{x+ay}{x-ay}-\frac{x-ay}{x+ay}+2\,\frac{x^2+a^2y^2}{x^2-a^2y^2}$.

61. $\frac{x^2-2x}{x^2-1}-\frac{x+3}{x+1}-\frac{4x}{1-x}$. 62. $\frac{1}{1-x}+\frac{1}{x+1}+\frac{1+2x-x^2}{1-x^2}$.

63. $\frac{1}{x+3y}+\frac{6y}{x^2-9y^2}-\frac{1}{3y-x}$.

64. $\frac{x^2-(y-z)^2}{(x+y)^2-z^2}+\frac{y^2-(z-x)^2}{(y+z)^2-x^2}+\frac{z^2-(x-y)^2}{(z+x)^2-y^2}$.

65. $\frac{a^2-(b+c)^2}{(a+b)^2-c^2}+\frac{b^2-(c+a)^2}{(b-c)^2-a^2}+\frac{c^2-(a-b)^2}{(c-a)^2-b^2}$.

134. **Multiplication of Fractions.** We have now to shew how to multiply any two algebraical fractions.

Let the fractions be $\frac{a}{b}$ and $\frac{c}{d}$.

Let $$x = \frac{a}{b} \times \frac{c}{d}.$$

Multiply by $b \times d$; then

$$x \times b \times d = \frac{a}{b} \times \frac{c}{d} \times b \times d$$

$$= \frac{a}{b} \times b \times \frac{c}{d} \times d, \qquad \text{by Art. 39.}$$

But $$\frac{a}{b} \times b = a, \text{ and } \frac{c}{d} \times d = c;$$

$$\therefore\ xbd = ac.$$

Divide by bd; then $x = \frac{ac}{bd}$.

That is $$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

Thus *the product of any two algebraical fractions is another fraction whose numerator is the product of their numerators, and whose denominator is the product of their denominators.*

The product of any number of fractions is found by the same rule.

For example:

$$\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ac}{bd} \times \frac{e}{f} = \frac{ace}{bdf}.$$

Again $$\frac{a}{b} \times \frac{c}{d} \times \frac{d}{a} = \frac{acd}{bda} = \frac{c}{b}.$$

And $$\frac{x+1}{x+2} \times \frac{x+2}{x+3} \times \frac{x+3}{x+1} = \frac{(x+1)(x+2)(x+3)}{(x+2)(x+3)(x+1)} = 1.$$

Def. Two fractions which are such that the numerator of each is the denominator of the other are called **reciprocal fractions.**

Thus $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$, and $\frac{1}{a}$ is the reciprocal of $\frac{a}{1}$ or a.

135. **Division of Fractions.** Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two fractions, and let $x = \frac{a}{b} \div \frac{c}{d}$.

Then $$x \times \frac{c}{d} = \frac{a}{b} \div \frac{c}{d} \times \frac{c}{d} = \frac{a}{b};$$

$$\therefore \; x \times \frac{c}{d} \times \frac{d}{c} = \frac{a}{b} \times \frac{d}{c}.$$

But $$x \times \frac{c}{d} \times \frac{d}{c} = x;$$

$$\therefore x = \frac{a}{b} \times \frac{d}{c}.$$

Thus *to divide by any fraction is the same as to multiply by its reciprocal.*

Ex. 1. Divide $\frac{x^2}{a^2}$ by $\frac{x}{a}$.

$$\frac{x^2}{a^2} \div \frac{x}{a} = \frac{x^2}{a^2} \times \frac{a}{x} = \frac{x}{a}.$$

Ex. 2. Divide $\frac{a}{b}$ by ab.

$$\frac{a}{b} \div ab = \frac{a}{b} \times \frac{1}{ab} = \frac{1}{b^2}.$$

Ex. 3. Divide $\frac{x-a}{x^3+a^3}$ by $\frac{x^3-a^3}{x+a}$.

$$\frac{x-a}{x^3+a^3} \div \frac{x^3-a^3}{x+a} = \frac{x-a}{x^3+a^3} \times \frac{x+a}{x^3-a^3}$$

$$= \frac{1}{(x^2-ax+a^2)(x^2+ax+a^2)} = \frac{1}{(x^4+a^2x^2+a^4)}.$$

EXAMPLES. XXXIX.

Reduce the following fractions to their simplest forms:

1. $\frac{2a}{3c} \times \frac{3c}{4a}$.

2. $\frac{5a^2}{6bc} \times \frac{3b^2}{10ca}$.

3. $\frac{a}{c} \div \frac{c}{d}$.

4. $\frac{2a^2}{bc} \div \frac{3ab}{c^2}$.

5. $\frac{a}{b} \times \frac{c}{d} \times \frac{bd}{ac}$.

6. $\frac{a^2}{bc} \times \frac{b^2}{ca} \times \frac{c^2}{ab}$.

7. $\frac{b}{a} \times \frac{b}{c} \div \frac{c}{a}$.

8. $\frac{a^2}{b^2} \times \frac{b^2}{c^2} \div \frac{c^2}{a^2}$.

9. $\frac{2a}{bc} \times \frac{2b}{ca} \times \frac{2c}{ab}$.

10. $\frac{2x^3}{3yz} \times \frac{3y^3}{5zx} \times \frac{5z^3}{2xy}$.

11. $\frac{3axy^3}{5b^2} \div \frac{6ay^3}{10b^2x}$.

12. $\frac{5a^2bc}{3b^2c^2a} \div \frac{5a^2bc^2}{3ab^2c^3}$.

13. By what must $\frac{3a^2b}{5x^2y}$ be multiplied, that the product may be $\frac{2a}{5x}$?

14. By what must $\frac{x^2y}{a^2z^2}$ be divided, that the quotient may be $\frac{b^2x}{a^2z}$?

Express in their simplest forms:

15. $\frac{x-y}{x^2+xy} \times \frac{x+y}{xy-y^2}$.

16. $\frac{a+b}{a^3-a^2b} \times \frac{ab-b^2}{ab+a^2}$.

17. $\frac{x^2+2x}{x^2-9} \times \frac{x^2-3x}{x^2-4}$.

18. $\frac{x^2-y^2}{x^2-4y^2} \times \frac{x-2y}{x+y}$.

19. $\frac{x^3+3x^2}{x+4} \div \frac{x+3}{x^2+4x}$.

20. $\frac{a+4b}{a^2+5ab} \div \frac{ab+4b^2}{a^3+5a^2b}$.

21. $\frac{x-1}{x-2} \times \frac{x-2}{x-3} \times \frac{x-3}{x-4}$.

22. $\frac{x+1}{(x+2)^2} \times \frac{x+2}{(x+3)^2} \times \frac{x+3}{(x+1)^2}$.

23. $\frac{x^2-3x+2}{x^2-5x+6} \times \frac{x^2-7x+12}{x^2-5x+4}$.

24. $\frac{x^2-1}{x^2+3x-10} \times \frac{x^2-25}{x^2-3x-4}$.

25. $\frac{x^3-a^3}{x^2-4a^2} \times \frac{x+2a}{x-a}$.

26. $\frac{a^3-x^3}{a^3+x^3} \div \frac{(a-x)^2}{a^2-x^2}$.

27. $\frac{a+x}{(a-x)^2} \times \frac{a^2-x^2}{a^2+x^2} \times \frac{a^4-x^4}{(a+x)^3}$.

28. $\frac{(a+b)^2}{(a-b)^3} \times \frac{(a^2-b^2)^2}{a^4-b^4} \div \frac{(a+b)^3}{a^2+b^2}$.

29. $\frac{(a-b)^2-c^2}{(a-c)^2-b^2} \times \frac{b^2-(c-a)^2}{c^2-(a-b)^2}$.

30. $\frac{x^2-(y+z)^2}{x^2-(y-z)^2} \div \frac{y^2-(x+z)^2}{y^2-(x-z)^2}$.

31. $\frac{x^6+y^6}{x^6-y^6} \times \frac{x-y}{x+y} \div \frac{x^4-x^2y^2+y^4}{x^4+x^2y^2+y^4}$.

32. $\left(1-\frac{2xy}{x^2+y^2}\right) \div \left(\frac{x^3-y^3}{x-y}-3xy\right)$.

136. We will now give some examples of more complex fractional expressions.

Ex. 1. Simplify $\dfrac{\dfrac{a}{b}}{\dfrac{c}{d}}$.

$$\frac{\dfrac{a}{b}}{\dfrac{c}{d}}=\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}.$$

Ex. 2. Simplify $\dfrac{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}$.

$$\frac{a+x}{a-x}-\frac{a-x}{a+x}=\frac{(a+x)(a+x)-(a-x)(a-x)}{(a-x)(a+x)}=\frac{4ax}{a^2-x^2}$$

and $$\frac{a+x}{a-x}+\frac{a-x}{a+x}=\frac{(a+x)(a+x)+(a-x)(a-x)}{(a-x)(a+x)}=\frac{2a^2+2x^2}{a^2-x^2}.$$

Hence the given fraction is equal to

$$\frac{4ax}{a^2-x^2}\div\frac{2a^2+2x^2}{a^2-x^2}=\frac{4ax}{a^2-x^2}\times\frac{a^2-x^2}{2a^2+2x^2}=\frac{2ax}{a^2+x^2}.$$

Ex. 3. Simplify $\dfrac{x}{x-\dfrac{x+2}{x+2-\dfrac{x+1}{x}}}$.

$$\frac{x}{x-\dfrac{x+2}{x+2-\dfrac{x+1}{x}}}=\frac{x}{x-\dfrac{(x+2)x}{(x+2)x-(x+1)}}$$

$$=\frac{x}{x-\dfrac{x^2+2x}{x^2+x-1}}=\frac{x(x^2+x-1)}{x(x^2+x-1)-(x^2+2x)}$$

$$=\frac{x(x^2+x-1)}{x^3-3x}=\frac{x^2+x-1}{x^2-3}.$$

Ex. 4. Simplify $\dfrac{\dfrac{a-b}{1+ab}+\dfrac{b-c}{1+bc}}{1-\dfrac{(a-b)(b-c)}{(1+ab)(1+bc)}}$.

$$\frac{a-b}{1+ab}+\frac{b-c}{1+bc}=\frac{(a-b)(1+bc)+(b-c)(1+ab)}{(1+ab)(1+bc)}$$

$$=\frac{a-b+abc-b^2c+b-c+ab^2-abc}{(1+ab)(1+bc)}$$

$$=\frac{a-c+ab^2-cb^2}{(1+ab)(1+bc)},$$

$$1-\frac{(a-b)(b-c)}{(1+ab)(1+bc)}=\frac{(1+ab)(1+bc)-(a-b)(b-c)}{(1+ab)(1+bc)}$$

$$=\frac{1+ab+bc+ab^2c-ab+b^2+ac-bc}{(1+ab)(1+bc)}$$

$$=\frac{1+ac+b^2+ab^2c}{(1+ab)(1+bc)}.$$

Hence the given fraction is equal to

$$\frac{a-c+ab^2-cb^2}{(1+ab)(1+bc)}\div\frac{1+ac+b^2+ab^2c}{(1+ab)(1+bc)}$$

$$=\frac{a-c+ab^2-cb^2}{1+ac+b^2+acb^2}=\frac{(a-c)(1+b^2)}{(1+ac)(1+b^2)}=\frac{a-c}{1+ac}.$$

EXAMPLES. XL.

Simplify

1. $\left\{\dfrac{x+2a}{a-2x}-\dfrac{a+2x}{x-2a}\right\}\times\left\{\dfrac{3}{2a-x}-\dfrac{1}{a-x}\right\}$.

2. $\left\{\dfrac{x}{1+x}+\dfrac{1-x}{x}\right\}\div\left\{\dfrac{x}{1+x}-\dfrac{1-x}{x}\right\}$.

3. $\left(\dfrac{x^2}{y}-\dfrac{y^2}{x}\right)\left(\dfrac{x}{y^2}+\dfrac{y}{x^2}\right)\div\left(\dfrac{x^2}{y}+\dfrac{y^2}{x}\right)\left(\dfrac{x}{y^2}-\dfrac{y}{x^2}\right)$.

4. $\dfrac{2}{\dfrac{1}{x}-\dfrac{1}{y}}+\dfrac{y}{1-\dfrac{y}{x}}-\dfrac{x}{1-\dfrac{x}{y}}$.

5. $\dfrac{x}{1+\dfrac{x}{y}}+\dfrac{y}{1+\dfrac{y}{x}}-\dfrac{2}{\dfrac{1}{x}+\dfrac{1}{y}}$.

6. $\dfrac{\dfrac{x}{y}+\dfrac{y}{x}+2}{x+y}+\dfrac{\dfrac{x}{y}+\dfrac{y}{x}-2}{x-y}$.

7. $\dfrac{\dfrac{a+b}{a+2b}+\dfrac{b}{a}}{\dfrac{a+b}{a}-\dfrac{b}{a+2b}}$.

8. $\dfrac{x^2-x+\dfrac{x-1}{x+1}}{x+\dfrac{1}{x+1}}$.

9. $\dfrac{\dfrac{x+2}{2x+3}-\dfrac{4x+5}{5x+6}}{\dfrac{2x+3}{3x+4}-\dfrac{3x+4}{4x+5}}$.

10. $\dfrac{\dfrac{1+x}{1+x^2}-\dfrac{1+x^2}{1+x^3}}{\dfrac{1+x^2}{1+x^3}-\dfrac{1+x^3}{1+x^4}}$.

11. $\dfrac{1}{x}-\dfrac{1}{x+\dfrac{1}{3x-\dfrac{1}{x}}}$.

12. $x^3-\dfrac{x^2}{1+\dfrac{1-x}{x+\dfrac{1}{x}}}$.

13. $\dfrac{1}{x+\dfrac{1}{1+\dfrac{a+1}{3-a}}}$.

14. $\dfrac{1}{x^2-\dfrac{x^3+1}{x+\dfrac{1}{x-1}}}$.

137. The following theorems are of importance.

Theorem I. *If the fractions* $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$, *&c. be all equal to one another, then will each be equal to the fraction*

$$\frac{pa+qc+re+\ldots}{pb+qd+rf+\ldots}.$$

Let each of the equal fractions be equal to x.

Then, since $\frac{a}{b}=x,\ \frac{c}{d}=x,\ \frac{e}{f}=x,$ &c.;

$$\therefore \quad a=bx, \quad c=dx, \quad e=fx, \quad \text{\&c.};$$

$$\therefore \quad pa=pbx, \quad qc=qdx, \quad re=rfx, \quad \text{\&c.}$$

Hence, by addition,

$$pa+qc+re+\ldots=pbx+qdx+rfx+\ldots=(pb+qd+rf+\ldots)\,x;$$

$$\therefore \quad \frac{pa+qc+re+\ldots}{pb+qd+rf+\ldots}=x=\frac{a}{b}=\text{\&c.}$$

As a particular case, each of the equal fractions $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, &c. is equal to $\frac{a+c+e+\dots}{b+d+f+\dots}$.

Theorem II. *If the denominators of the fractions* $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, *&c. be all positive, then will the fraction* $\frac{a+c+e+\dots}{b+d+f+\dots}$ *be greater than the least, and less than the greatest, of the fractions* $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, *&c.*

Let $\frac{a}{b}$ be the greatest of the fractions, and let $\frac{a}{b}=x$.

Then $\frac{c}{d}<x$, $\frac{e}{f}<x$, &c.

Hence, as b, d, f,… are all positive, we have

$$a=bx, \quad c<dx, \quad e<fx, \text{ \&c.}$$

Hence, by addition,

$$a+c+e+\dots<bx+dx+fx+\dots<(b+d+f+\dots)x;$$

$$\therefore \quad \frac{a+c+e+\dots}{b+d+f+\dots}<x.$$

Hence $\frac{a+c+e+\dots}{b+d+f+\dots}$ is less than the greatest of the fractions; and it can be similarly proved to be greater than the least of the fractions.

Ex. 1. Shew that, if $\frac{a}{b}=\frac{c}{d}$, then will $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

Let $\frac{a}{b}=x$; then $\frac{c}{d}=x$. Hence $a=bx$, and $c=dx$.

Hence $$\frac{a+b}{a-b}=\frac{bx+b}{bx-b}=\frac{x+1}{x-1}.$$

Also $$\frac{c+d}{c-d}=\frac{dx+d}{dx-d}=\frac{x+1}{x-1}.$$

And, since $\frac{a+b}{a-b}$ and $\frac{c+d}{c-d}$ are both equal to $\frac{x+1}{x-1}$, they must be equal to one another.

Ex. 2. Shew, that if $\frac{a}{b}=\frac{c}{d}$ then will $\frac{a^2+b^2}{c^2+d^2}=\frac{a^2-b^2}{c^2-d^2}$.

Let $\frac{a}{b}=x$; then $\frac{c}{d}=x$. Hence $a=bx$, and $c=dx$.

Hence $$\frac{a^2+b^2}{c^2+d^2}=\frac{b^2x^2+b^2}{d^2x^2+d^2}=\frac{b^2(x^2+1)}{d^2(x^2+1)}=\frac{b^2}{d^2}.$$

Also $$\frac{a^2-b^2}{c^2-d^2}=\frac{b^2x^2-b^2}{d^2x^2-d^2}=\frac{b^2(x^2-1)}{d^2(x^2-1)}=\frac{b^2}{d^2}.$$

Hence $$\frac{a^2+b^2}{c^2+d^2}=\frac{a^2-b^2}{c^2-d^2}.$$

Ex. 3. Shew that, if $\frac{cy+bz}{l}=\frac{az+cx}{m}=\frac{bx+ay}{n}$,

then will $$\frac{bcx}{-al+bm+cn}=\frac{cay}{al-bm+cn}=\frac{abz}{al+bm-cn}.$$

By Theorem I, each of the given equal fractions is equal to
$$\frac{-a(cy+bz)+b(az+cx)+c(bx+ay)}{-al+bm+cn}=\frac{2bcx}{-al+bm+cn}\ldots$$

And similarly each of the fractions can be proved to be equal to $\frac{2cay}{al-bm+cn}$ or to $\frac{2abz}{al+bm-cn}$.

EXAMPLES. XLI.

Simplify

1. $\frac{6x^4y^6z^7}{9x^2y^6z^9}$.

2. $\frac{18a^2b^6c^7x}{24ab^6c^8x^2}$.

3. $\frac{a^5b^7c^9x^3y^5z^7}{a^3b^5c^7x^5y^7z^9}$.

4. $\frac{x^2-6xy+5y^2}{x^3-5x^2y+4xy^2}$.

5. $\frac{x^3-10x^2y-11xy^2}{x^2y-8xy^2-9y^3}$.

6. $\frac{(a^2-b^2)(a^3-b^3)}{(a-b)(a^4-b^4)}$.

7. $\frac{(a^2-b^2)(a^6-b^6)}{(a+b)^2(a-b)^2}$.

8. $\frac{x^3+3x^2-x-3}{x^4+4x^3-12x-9}$.

9. $\frac{3x^4+5x^3-7x^2+2x+2}{2x^4+3x^3-2x^2+12x+5}$.

10. $\frac{2x^4+9x^3y+14xy^3+3y^4}{3x^4+14x^3y+9xy^3+2y^4}$.

11. $\frac{2x^3-11x^2y+11xy^2+4y^3}{2x^4-3x^3y+7x^2y^2-12xy^3-4y^4}$.

12. $2\frac{x+y}{3}-3\frac{x-y}{5}$.

13. $5\frac{x-2y}{4}-4\frac{y-2x}{5}$.

14. $\dfrac{1}{4-x}+\dfrac{4}{x^2-16}$. 15. $\dfrac{1}{3-x}+\dfrac{3}{x^2-9}$.

16. $\dfrac{x}{x-2y}-\dfrac{x^2}{(2y-x)^2}$. 17. $\dfrac{a}{a-4b}+\dfrac{4ab}{(4b-a)^2}$.

18. $\dfrac{1}{a+4}-\dfrac{1}{a+5}+\dfrac{1}{a+6}-\dfrac{1}{a+7}$. 19. $\dfrac{1}{a}-\dfrac{3}{a+3}+\dfrac{3}{a+6}-\dfrac{1}{a+9}$.

20. $\dfrac{x^3-y^3}{x^2-9y^2}\times\dfrac{x+3y}{x-y}$. 21. $\dfrac{x^2-y^2}{x^2-4y^2}\div\dfrac{y-x}{2y-x}$.

22. $\left\{\left(\dfrac{1}{a}+\dfrac{1}{b}\right)^2-\dfrac{1}{ab}-\dfrac{4}{(a-b)^2}\right\}\div\left\{\left(\dfrac{1}{a}-\dfrac{1}{b}\right)^2-\dfrac{1}{ab}\right\}$.

23. $\left(\dfrac{x}{x-2}+\dfrac{5}{x-8}\right)\times\left(\dfrac{x-3}{3x-8}-\dfrac{2}{x+2}\right)$.

24. $\dfrac{(x-y)^4-xy\,(x-y)^2-2x^2y^2}{(x-y)\,(x^3-y^3)+2x^2y^2}$. 25. $\dfrac{1}{x-1}+\dfrac{2x+1}{x^2+x+1}-\dfrac{3}{x}$.

26. $\dfrac{1}{6a-2b}+\dfrac{1}{3a+2b}-\dfrac{3}{6a+2b}$. 27. $\dfrac{1}{8a+2b}-\dfrac{1}{2b-8a}+\dfrac{b}{16a^2-b^2}$.

28. $\dfrac{a^6+b^6}{a^6-b^6}\times\dfrac{a-b}{a^2+b^2}\div\dfrac{a^4-a^2b^2+b^4}{a^4+a^2b^2+b^4}$. 29. $\dfrac{\dfrac{m^2+n^2}{n}-m}{\dfrac{1}{n}-\dfrac{1}{m}}\times\dfrac{m^2-n^2}{m^3+n^3}$.

30. $\dfrac{\left\{1+\dfrac{c}{a+b}+\dfrac{c^2}{(a+b)^2}\right\}\left\{1-\dfrac{c^2}{(a+b)^2}\right\}}{\left\{1-\dfrac{c^3}{(a+b)^3}\right\}\left\{1+\dfrac{c}{a+b}\right\}}$.

31. $\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}-\dfrac{4y^2}{x^2-y^2}$. 32. $\left\{1+\dfrac{2b^2}{a\,(a-3b)}\right\}\left\{1+\dfrac{b}{2b-a}\right\}$.

33. $\dfrac{x^3+y^3}{xy\,(y^2-x^2)}-\dfrac{y}{x^2+xy}+\dfrac{x}{xy-y^2}$.

34. $\dfrac{a+b}{a+5b}+\dfrac{a^2-b^2}{25b^2-a^2}-\dfrac{b-a}{a-5b}$. 35. $\dfrac{a-b}{a-3b}+\dfrac{a+b}{a+3b}-\dfrac{b^2-a^2}{9b^2-a^2}$.

36. $\dfrac{1}{2\,(x-1)}-\dfrac{x-5}{x^2-7x+10}+\dfrac{1}{2}\,\dfrac{x-6}{x^2-9x+18}$.

37. $\dfrac{3}{x-3}-\dfrac{2}{x-4}-\dfrac{x-6}{(x-2)\,(x-5)}$.

38. $\dfrac{3}{x-4}-\dfrac{2}{x-5}-\dfrac{x-7}{(x-6)\,(x-3)}$.

39. $$\left\{x-y-\frac{1}{x-y+\dfrac{xy}{x-y}}\right\}\times\frac{x^3+y^3}{x^2-y^2}.$$

40. $$\left\{x+y-\frac{1}{x+y-\dfrac{xy}{x+y}}\right\}\times\frac{x^3-y^3}{x^2-y^2}.$$

41. $$\left\{1-\frac{1-x}{1+x}+\frac{1+2x^2}{1-x^2}\right\}\times\frac{x+1}{2x+1}.$$

42. $$\frac{x-2-\dfrac{1}{x-2}}{x-2-\dfrac{4}{x-5}}\times\frac{x-4-\dfrac{4}{x-4}}{x-4-\dfrac{1}{x-4}}.$$

43. $$\frac{6}{x^2+2x-8}-\frac{7}{x^2+x-12}+\frac{1}{x^2-5x+6}.$$

44. $$(x^3+y^3)\left\{\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}\right\}\times\frac{1}{x^2-xy+y^2}\div\left\{\frac{x+y}{x-y}-\frac{x-y}{x+y}\right\}.$$

45. $$\left(\frac{a^2+b^2}{2ab}-1\right)\frac{ab^2}{a^3+b^3}\div\frac{4ab\,(a+b)}{a^2-ab+b^2}.$$

46. $$\left(\frac{a}{b}+\frac{b}{a}\right)^2+\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2-\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right).$$

47. $$\left\{\frac{1}{3x^2-14xy+15y^2}+\frac{2}{3x^2-2xy-5y^2}\right\}\div\left\{\frac{x+y}{x-3y}-\frac{x-y}{x+3y}\right\}.$$

48. $$\frac{b-c}{a+x}+\frac{c-a}{b+x}+\frac{a-b}{c+x}+\frac{(b-c)(c-a)(a-b)}{(a+x)(b+x)(c+x)}.$$

49. $$\left\{\left(\frac{a-b}{a+b}\right)^2-2\,\frac{a-b}{a+b}+1\right\}\div\left\{\left(\frac{a+b}{a-b}\right)^2-2\,\frac{a+b}{a-b}+1\right\}.$$

50. $$\left\{\left(\frac{a-b}{a+b}\right)^3+3\left(\frac{a-b}{a+b}\right)^2+3\,\frac{a-b}{a+b}+1\right\}$$
$$\div\left\{\left(\frac{a+b}{a-b}\right)^3+3\left(\frac{a+b}{a-b}\right)^2+3\,\frac{a+b}{a-b}+1\right\}.$$

51. $$\frac{16}{x-17}-\frac{21}{x-9}-\frac{16}{x+4}+\frac{21}{x+7}.$$

52. $$\frac{11}{x-5}-\frac{7}{x-7}-\frac{11}{x+2}+\frac{7}{x+4}.$$

53. $$\frac{x+7}{x+1}+\frac{x+8}{x+2}-\frac{x+9}{x+3}-\frac{x+10}{x+4}.$$

54. $\dfrac{(a+b-c)^2-d^2}{(a+b)^2-(c+d)^2}+\dfrac{(b+c-a)^2-d^2}{(b+c)^2-(a+d)^2}+\dfrac{(c+a-b)^2-d^2}{(c+a)^2-(b+d)^2}.$

55. Simplify $\dfrac{1}{axy}-\dfrac{1}{a(x-a)(y-a)}+\dfrac{1}{y(x-a)(y-a)}.$

56. Simplify $\dfrac{a+x}{x(x-y)(x-z)}+\dfrac{a+y}{y(y-z)(y-x)}+\dfrac{a+z}{z(z-x)(z-y)}.$

57. Find the value of $\dfrac{a-2x}{b-2x}$ when $x=\dfrac{ab}{a+b}$.

58. Find the value of

$$\frac{a+b+2c}{a+b-2c}+\frac{a+b+2d}{a+b-2d},$$

when $$a+b=\frac{4cd}{c+d}.$$

59. If $\dfrac{a}{b}=\dfrac{c}{d}$, prove (i) $\dfrac{a-b}{a-2b}=\dfrac{c-d}{c-2d}$. (ii) $\dfrac{a^2}{(a+b)^2}=\dfrac{c^2}{(c+d)^2}$.

(iii) $\dfrac{a^2-b^2}{a^2+b^2}=\dfrac{c^2-d^2}{c^2+d^2}$. (iv) $\dfrac{pa^2+qab+rb^2}{la^2+mab+nb^2}=\dfrac{pc^2+qcd+rd^2}{lc^2+mcd+nd^2}$.

60. If $\dfrac{x}{a-b}=\dfrac{y}{b-c}=\dfrac{z}{c-a}$; then will $x+y+z=0$.

61. If $\dfrac{y+z}{ay+bz}=\dfrac{z+x}{az+bx}=\dfrac{x+y}{ax+by}$; then will each fraction $=\dfrac{2}{a+b}$.

62. Prove that, if a, b, c be unequal, and $\dfrac{b-c}{x}=\dfrac{c-a}{y}=\dfrac{a-b}{z}$, then will $x+y+z=0$ and $ax+by+cz=0$.

63. Prove that, if $\dfrac{a+b}{a-b}=\dfrac{b+c}{2(b-c)}=\dfrac{c+a}{3(c-a)}$,

then $$8a+9b+5c=0.$$

64. Prove that, if $\dfrac{bz-cy}{b-c}=\dfrac{cx-az}{c-a}$, then each is equal to $\dfrac{ay-bx}{a-b}$.

65. Prove that, if

$$\frac{a}{2y+2z-3x}=\frac{b}{2z+2x-3y}=\frac{c}{2x+2y-3z},$$

then $$\frac{x}{a+2b+2c}=\frac{y}{b+2c+2a}=\frac{z}{c+2a+2b}.$$

CHAPTER XIII.

EQUATIONS WITH FRACTIONS.

138. In the present chapter we shall give examples of equations which contain fractional expressions.

Ex. 1. Solve $$\frac{x-1}{5}-\frac{3x-1}{8}=\frac{43-5x}{6}.$$

We may multiply both sides of the equation by 120, the L.C.M. of the denominators of the fractions, without destroying the equality: we thus get rid of fractions, and have

$$24(x-1)-15(3x-1)=20(43-5x);$$

$$\therefore \quad 24x-24-45x+15=860-100x.$$

Transposing, we have

$$24x-45x+100x=860+24-15;$$

$$\therefore \quad 79x=869.$$

Hence $$x=\frac{869}{79}=11.$$

Ex. 2. Solve $$\frac{1}{x-3}+\frac{2}{x+3}=\frac{3}{x+5}.$$

The L.C.M. of the denominators is $(x-3)(x+3)(x+5)$. Multiplying both sides of the equation by the L.C.M.[1], we get rid of fractions, and have

$$(x+3)(x+5)+2(x-3)(x+5)=3(x-3)(x+3);$$

$$\therefore \quad x^2+8x+15+2x^2+4x-30=3x^2-27.$$

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[1] See Art. 148.

By transposition, we have

$$x^2+2x^2-3x^2+8x+4x=-27-15+30;$$

$$\therefore \qquad 12x=-12,$$

or

$$x=-1.$$

Ex. 3. Solve $\dfrac{1}{x+1}+\dfrac{1}{x+7}=\dfrac{1}{x+3}+\dfrac{1}{x+5}.$

In this case it is best not to multiply at once by the L.C.M. of the demominators: the work is simplified by proceeding as under.

We have

$$\frac{1}{x+1}-\frac{1}{x+3}+\frac{1}{x+7}-\frac{1}{x+5}=0;$$

$$\therefore \qquad \frac{x+3-(x+1)}{(x+1)(x+3)}+\frac{x+5-(x+7)}{(x+5)(x+7)}=0,$$

that is

$$\frac{2}{(x+1)(x+3)}-\frac{2}{(x+5)(x+7)}=0;$$

$$\therefore \qquad (x+1)(x+3)=(x+5)(x+7),$$

that is

$$x^2+4x+3=x^2+12x+35;$$

$$\therefore \qquad 4x-12x=35-3;$$

$$\therefore \qquad -8x=32,$$

$$\therefore \qquad x=32\div(-8)=-4.$$

Ex. 4. Solve the equation

$$\frac{x-1}{x+1}+\frac{x+5}{x+7}=\frac{x+1}{x+3}+\frac{x+3}{x+5}.$$

Since

$$\frac{x-1}{x+1}=1-\frac{2}{x+1}, \quad \frac{x+5}{x+7}=1-\frac{2}{x+7}, \quad \frac{x+1}{x+3}=1-\frac{2}{x+3},$$

and

$$\frac{x+3}{x+5}=1-\frac{2}{x+5},$$

we have

$$1-\frac{2}{x+1}+1-\frac{2}{x+7}=1-\frac{2}{x+3}+1-\frac{2}{x+5};$$

$$\therefore \qquad \frac{2}{x+1}+\frac{2}{x+7}=\frac{2}{x+3}+\frac{2}{x+5},$$

which is the same equation as that in Ex. 3.

Ex. 5. Solve the equation

$$\frac{4x+5}{x+1}+\frac{x+5}{x+4}=\frac{2x+5}{x+2}-\frac{x^2-10}{x+3}+x.$$

By division we have

$$\frac{4x+5}{x+1}=4+\frac{1}{x+1},\quad \frac{x+5}{x+4}=1+\frac{1}{x+4},$$

$$\frac{2x+5}{x+2}=2+\frac{1}{x+2} \text{ and } \frac{x^2-10}{x+3}=x-3-\frac{1}{x+3}.$$

Hence $$4+\frac{1}{x+1}+1+\frac{1}{x+4}=2+\frac{1}{x+2}-\left(x-3-\frac{1}{x+3}\right)+x;$$

$$\therefore \quad \frac{1}{x+1}+\frac{1}{x+4}=\frac{1}{x+2}+\frac{1}{x+3};$$

$$\therefore \quad \frac{1}{x+1}-\frac{1}{x+2}=\frac{1}{x+3}-\frac{1}{x+4};$$

$$\therefore \quad \frac{(x+2)-(x+1)}{(x+1)(x+2)}=\frac{(x+4)-(x+3)}{(x+4)(x+3)},$$

that is $$\frac{1}{(x+1)(x+2)}=\frac{1}{(x+4)(x+3)};$$

$$\therefore \quad (x+1)(x+2)=(x+4)(x+3);$$

that is $$x^2+3x+2=x^2+7x+12;$$

$$\therefore \quad x^2+3x-x^2-7x=12-2;$$

$$\therefore \quad -4x=10;$$

$$\therefore \quad x=-\frac{10}{4}=-\frac{5}{2}.$$

EXAMPLES. XLII.

1. $\frac{x-2}{5}-\frac{x-3}{4}=\frac{x-7}{10}.$

2. $\frac{3x-1}{11}-2\frac{x-1}{5}=\frac{2-x}{10}.$

3. $\frac{x}{6}-\frac{x-2}{5}=\frac{x}{5}-3\frac{x-3}{4}.$

4. $\frac{x-2}{3}-3\frac{x-1}{5}=x-5\frac{x-2}{6}.$

5. $\frac{1-3x}{2}+\frac{3x+1}{2}=\frac{2}{1-3x}.$

6. $\frac{3-4x}{6}-\frac{5-8x}{12}=\frac{1-x}{1+x}.$

7. $\frac{3x-1}{4}-2\frac{x+1}{x-3}=2-\frac{3-6x}{8}.$

8. $\frac{2x}{5}-3\frac{x-1}{x+1}=3-\frac{1-4x}{10}.$

9. $\frac{1}{4x+6}+\frac{1}{6x+4}=\frac{2}{2x+3}$.

10. $\frac{1}{3x+9}+\frac{2}{5x+1}=\frac{2}{x+3}$.

11. $\frac{1}{4x+12}-\frac{3}{4x+1}=\frac{4}{2x+6}$.

12. $\frac{3}{2x+3}+\frac{5}{4x+6}=\frac{7}{6x+8}$.

13. $\frac{4x}{x+1}-\frac{x}{x-2}=3$.

14. $\frac{3x}{x+6}-\frac{x}{x+5}=2$.

15. $\frac{6x}{x-7}-\frac{x}{x-6}=5$.

16. $\frac{2x}{x+3}-\frac{4x}{x+7}+2=0$.

17. $\frac{1}{x+4}+\frac{2}{x+6}=\frac{3}{x+5}$.

18. $\frac{3}{x+1}-\frac{2}{x+2}=\frac{1}{x+3}$.

19. $\frac{5}{2}\,\frac{1}{x+4}-\frac{3}{2}\,\frac{1}{x+2}=\frac{1}{x+6}$.

20. $\frac{3}{2}\,\frac{1}{x+2}-\frac{5}{4}\,\frac{1}{x+3}-\frac{1}{4x}=0$.

21. $\frac{2x-5}{3x-7}=\frac{2x-7}{3x-5}$.

22. $\frac{6x-2}{8x-5}=\frac{3x+7}{4x+8}$.

23. $\frac{x-9}{3x-7}=\frac{2x-5}{6x-4}$.

24. $\frac{3-2x}{x-5}=\frac{2x-7}{4-x}$.

25. $\frac{x+1}{x-1}-\frac{x-3}{x+3}=\frac{8}{x}$.

26. $\frac{x+2}{x-2}-\frac{x-2}{x+2}=\frac{8}{x+1}$.

27. $\frac{x-1}{x+3}-\frac{x+2}{x-4}+\frac{10}{x}=0$.

28. $\frac{3x}{x+2}-3\,\frac{x-1}{x-2}+\frac{9}{x+1}=0$.

29. $3\,\frac{x-1}{x+1}+2\,\frac{x+1}{x-1}=5$.

30. $5\,\frac{x-2}{x+2}-2\,\frac{x-3}{x+3}=3$.

31. $\frac{8x}{x^3-1}=\frac{1}{x-4}-\frac{1}{x+4}$.

32. $\frac{12x}{x^3-9}+\frac{2}{x+3}=\frac{2}{x-3}$.

33. $\frac{4}{x^2-1}+\frac{1}{x+1}=\frac{1}{1-x}$.

34. $\frac{3}{x^2-9}+\frac{1}{x+3}=\frac{2}{3-x}$.

35. $\frac{2x+1}{2x-1}-\frac{8}{4x^2-1}=\frac{2x-1}{1+2x}$.

36. $\frac{3x+5}{3x-1}+\frac{5}{1-9x^2}=\frac{8+3x}{1+3x}$.

37. $\frac{1}{x-5}+\frac{3}{2x-6}=\frac{5}{(x-3)(x-5)}$.

38. $\frac{1}{2x-4}+\frac{3}{x-5}=\frac{7x+1}{(x-2)(x-5)}$.

39. $\frac{6}{2-3x}+\frac{10}{6-5x}+\frac{4}{10+x}=0$.

40. $\frac{6}{3x-1}+\frac{7}{3-7x}-\frac{1}{5+x}=0.$

41. $\frac{1}{x+5}-\frac{1}{x+6}=\frac{1}{x+6}-\frac{1}{x+8}.$

42. $\frac{1}{x+7}-\frac{1}{x+1}=\frac{1}{x+1}-\frac{1}{x+3}.$

43. $\frac{1}{x+2}+\frac{1}{x+10}=\frac{1}{x+4}+\frac{1}{x+8}.$

44. $\frac{1}{x-2}-\frac{1}{x-6}=\frac{1}{x-4}-\frac{1}{x-8}.$

45. $\frac{1}{x-5}+\frac{1}{x+2}=\frac{1}{x-4}+\frac{1}{x+1}.$

46. $\frac{x}{x-2}+\frac{x-9}{x-7}=\frac{x+1}{x-1}+\frac{x-8}{x-6}.$

47. $\frac{2x+1}{x+1}+\frac{2x+9}{x+5}=\frac{2x+3}{x+2}+\frac{2x+7}{x+4}.$

48. $\frac{2x-3}{2x-4}-\frac{2x-4}{2x-5}=\frac{2x-7}{2x-8}-\frac{2x-8}{2x-9}.$

49. $\frac{x+7}{x+5}+\frac{x+9}{x+7}=\frac{x+6}{x+4}+\frac{x+10}{x+8}.$

50. $\frac{16x-13}{4x-3}+\frac{40x-43}{8x-9}=\frac{32x-30}{8x-7}+\frac{20x-24}{4x-5}.$

51. $\frac{x-7}{x-5}-\frac{x-8}{x-6}=\frac{2x-7}{2x-5}-\frac{2x-11}{2x-9}.$

52. $\frac{7}{x-9}-\frac{11}{x-4}=\frac{7}{x+2}-\frac{11}{x+3}.$

53. $\frac{16}{x-17}-\frac{21}{x-9}=\frac{16}{x+4}-\frac{21}{x+7}.$

54. $\frac{x+a}{x-b}+\frac{x+b}{x-a}=2.$

55. $\frac{x+a}{x-a}-\frac{x-b}{x+b}=\frac{2(a+b)}{x}.$

56. $\frac{ax}{a+x}+\frac{bx}{b+x}=a+b.$

57. $\frac{a+c}{x+2b}+\frac{b+c}{x+2a}=\frac{a+b+2c}{x+a+b}.$

58. $\frac{x-b}{x-a}-\frac{x-a}{x-b}=\frac{2(a-b)}{x-a-b}.$

MISCELLANEOUS EXAMPLES. III.

A. 1. What must be added to $x^3-3x^2y+3xy^2$ that the sum may be

$$3x^2y-3xy^2+y^3?$$

2. Shew that

$$x(y+z)^2+y(z+x)^2+z(x+y)^2-4xyz=(y+z)(z+x)(x+y).$$

3. Find the factors of (i) x^4-y^4, (ii) $a^2-b^2+2a-2b$, and (iii) $x^2-3x-28$.

4. Find the H.C.F. of $x^4+9x-20$ and $5x^4+9x^3-64$.

Simplify $$\frac{x^4+9x-20}{5x^5+9x^4-64x}.$$

5. Solve

(i) $\dfrac{x-1}{2}+\dfrac{2x-7}{3}=x-2.$

(ii) $3x-4y+2=5x-6y-2=7x+2y+4.$

(iii) $\dfrac{a}{x+b}+\dfrac{b}{x+a}=\dfrac{a+b}{x}.$

6. If A were to give 10 shillings to B, he would then have three times as much as B; but if B were to give 5 shillings to A, A would have four times as much as B. How much has each?

B. 1. Find the value of $a(a+b)(a+b+c)-a(a-b)(a-b-c)$, when $a=5$, $b=3$ and $c=-6$; also when $a=-3$, $b=-2$, and $c=4$.

2. Shew that $a^2+3(a-2b)^2=3(a-b)^2+(a-3b)^2$, and that

$$\left(a+\frac{1}{a}\right)^2-\left(b+\frac{1}{b}\right)^2=\left(ab-\frac{1}{ab}\right)\left(\frac{a}{b}-\frac{b}{a}\right).$$

3. Find the factors of (i) x^3-4x^2+4x, (ii) $2x^2-5x+2$, and (iii) x^3-x^2+x-1.

4. Simplify $\dfrac{1}{x-3}+\dfrac{1}{x+3}-\dfrac{6}{x^2-9}$, and shew that

$$\left(y-\frac{a^2-xy}{y-x}\right)\left(x+\frac{a^2-xy}{y-x}\right)+\left(\frac{a^2-xy}{y-x}\right)^2=a^2.$$

5. Solve

(i) $$\frac{1}{x-1}+\frac{2}{x-2}=\frac{3}{x-3},$$

(ii) $$\left.\begin{aligned}\frac{x}{a}-\frac{y}{b}&=1\\ \frac{x}{b}+\frac{y}{a}&=\frac{a}{b}\end{aligned}\right\}.$$

6. Find the fraction which becomes $\frac{1}{5}$ when 2 is added to its numerator, and $\frac{1}{6}$ when 2 is taken from its denominator.

C. 1. Prove that

$$(x-1)^2(y^2+1)-(x^2+1)(y-1)^2=2(x-y)(xy-1),$$

and that

$$(x+y)(x+z)-x^2=(y+z)(y+x)-y^2=(z+x)(z+y)-z^2.$$

2. Prove that the product of any two numbers is equal to a quarter of the difference of the square of their sum and the square of their difference.

3. Find the factors of (i) $1+18x-63x^2$, (ii) $3x^3y-24y^4$, and (iii) $(x^3+3x)^2-(3x^2+1)^2$.

4. Simplify $$\frac{x^2-10x+21}{x^3-46x-21},$$

and $$\frac{1}{(x-3)(x-2)}-\frac{x-4}{(x-1)(x-3)}+\frac{x-3}{(x-1)(x-2)}.$$

5. Solve

(i) $$\frac{x-3}{2}-\frac{x-1}{4}+\frac{x+1}{6}-\frac{x-3}{8}=0,$$

(ii) $ax+by=c,\ a^2x+b^2y=c^2$.

6. *A* and *B* have £100 between them; but if *A* were to lose one-half of his money, and *B* were to lose one-third of his, they would then have only £55 between them. How much has each?

D. 1. Find the value of $\sqrt{a^3+b^3+c^3}-(a-b-c)^2$, when $a=3$, $b=-3$ and $c=4$.

2. Shew that

$$(x-5)(x-4)-3(x-2)(x-1)+3(x+1)(x+2)-(x+4)(x+5)=0,$$

and that

$$1+n+\tfrac{1}{2}n(n+1)+\tfrac{1}{6}n(n+1)(n+2)=\tfrac{1}{6}(n+1)(n+2)(n+3).$$

3. Find the factors of (i) $x^3+x^2y-6xy^2$, (ii) $x^3+ax^2-a^2x-a^3$ and (iii) $x^2y^2-x^2-y^2+1$.

4. Simplify

(i) $\dfrac{a-x}{a+x}+\dfrac{4ax}{a^2-x^2}+\dfrac{a+x}{a-x}$.

(ii) $\dfrac{1}{x-4}-\dfrac{2}{x-5}+\dfrac{1}{x-6}$.

5. Solve

(i) $\dfrac{3x+1}{4}-2(6-x)=\dfrac{5x-4}{7}-\dfrac{x-2}{3}$.

(ii) $3y+\dfrac{9}{x}+6=0,\ y+\dfrac{5}{x}=8$.

6. A hare is pursued by a greyhound, and is 60 of her own leaps before him. The hare takes 3 leaps in the time that the greyhound takes 2; but the greyhound goes as far in 3 leaps as the hare does in 7. In how many leaps will the greyhound catch the hare?

E. 1. Shew that

$$(m+n)(m+n-1)=m(m-1)+2mn+n(n-1),$$

and that

$$(m+n)(m+n-1)(m+n-2)=m(m-1)(m-2)$$
$$+3m(m-1)n+3mn(n-1)+n(n-1)(n-2).$$

2. Divide x^4-y^4 by $x-y$, and from the result *write down* the quotient when $(x+y)^4-16z^4$ is divided by $x+y-2z$.

3. Find the factors of

$$5x^2+24x-5, \text{ and of } a^3-2abc-ab^2-ac^2.$$

4. Find the L.C.M. of $8a^3-18ab^2$, $8a^3+8a^2b-6ab^2$ and

$$4a^2-8ab+3b^2.$$

5. Shew that

(i) $1+\dfrac{a}{x-a}+\dfrac{bx}{(x-a)(x-b)}=\dfrac{x^2}{(x-a)(x-b)}$,

(ii) $$1+\frac{a}{x-a}+\frac{bx}{(x-a)(x-b)}+\frac{cx^2}{(x-a)(x-b)(x-c)}$$
$$=\frac{x^3}{(x-a)(x-b)(x-c)}.$$

6. A debtor is just able to pay his creditors five shillings in the pound; but if his assets had been five times as great, and his debts two-thirds of what they really were, he would have had a balance in his favour of £140. How much did he owe?

F. 1. Find the value of $\{a-(b-c)\}^2+\{b-(c-a)\}^2+\{c-(a-b)\}^2$ when $a=-1$, $b=-3$ and $c=-5$.

2. Divide $x^8-2a^4x^4+a^8$ by $x^3+ax^2+a^2x+a^3$.

3. Find the L.C.M. of x^2-1, $(x-1)^2$, $(x+1)^2$, x^3-1 and x^3+1.

4. Add together $\dfrac{1}{2x+y}$, $\dfrac{1}{2x-y}$ and $\dfrac{3x}{y^2-4x^2}$; and shew that

$$\frac{(x-a)(y-a)}{(a-b)(a-c)}+\frac{(x-b)(y-b)}{(b-c)(b-a)}+\frac{(x-c)(y-c)}{(c-a)(c-b)}=1.$$

5. Solve

(i) $$\frac{1}{x+a}+\frac{1}{x+b}=\frac{2}{x+a+b},$$

(ii) $$\left.\begin{aligned} 5x+2y+3z&=13\\ 3x+7y-z&=2\\ x-2y+z&=5 \end{aligned}\right\}.$$

6. When the arable land of a farm was let at 30 shillings, and the pasture at 40 shillings an acre, the total rent of a farm was £550. When the rent of the pasture was reduced by 5 shillings an acre, and that of the arable land by 10 shillings an acre, the whole rent was £387. 10*s*. What was the total acreage of the farm?

CHAPTER XIV.

QUADRATIC EQUATIONS.

139. It is clear that a product is zero when any one of its factors is zero; and it is also clear that a product can never be zero unless one of its factors is zero. Thus ab is zero if a is zero, or if b is zero; and, if we know that ab is zero, we conclude that *either* a *or* b must be zero.

So also abc is zero if a, or b, or c is zero; and, if we know that abc is zero, we conclude that either a, *or* b, *or* c must be zero; and so on, however many factors there may be.

Similarly the product $(x-2)(x-4)$ is zero, if $x-2$ is zero, or if $x-4$ is zero; and, in order that the product may be zero, one of these factors must be zero.

Thus the equation

$$(x-2)(x-4)=0$$

is satisfied if $x-2=0$, or if $x-4=0$, that is, if $x=2$, or if $x=4$, and in no other case. The *roots* of the equation are therefore 2 and 4.

Again, the equation

$$(x-3)(x-4)(x-5)=0$$

is satisfied if $x-3=0$, or if $x-4=0$, or if $x-5=0$, and in no other case; so that in order that the equation may be true x must be either 3, 4 or 5: thus 3, 4 and 5 are the *roots* of the equation

$$(x-3)(x-4)(x-5)=0.$$

Thus the equation

$$(x-a)(x-b)(x-c)\ldots\ldots=0$$

is equivalent to the system of *alternative* equations

$$x-a=0, \text{ or } x-b=0, \text{ or } x-c=0, \text{ \&c.}$$

140. From the examples considered in the last article it will be apparent that *the solution of an equation of any degree can be written down at once provided the equation is given in the form of a product of factors of the first degree equated to zero.*

The following are examples of such equations:

Ex. 1. Solve $(x-1)(x+1)=0$.

The equation is satisfied if $x-1=0$, or if $x+1=0$, and in no other case.

Hence we must have

$$x-1=0, \text{ or } x+1=0,$$

that is $x=1$, or $x=-1$.

Thus the roots of the equation are 1 and -1.

Ex. 2. Solve $x(x+1)(x+2)=0$.

The equation is satisfied if $x=0$, or if $x+1=0$, or if $x+2=0$, and in no other case.

Hence we must have

$$x=0, \text{ or } x+1=0, \text{ or } x+2=0,$$

that is $x=0$, or $x=-1$, or $x=-2$.

Thus the roots of the equation are 0, -1 and -2.

Ex. 3. Solve $x(2x-1)(2x+3)=0$.

The equation is satisfied if $x=0$, or if $2x-1=0$, or if $2x+3=0$, and in no other case.

Hence we must have

$$x=0, \text{ or } 2x-1=0, \text{ or } 2x+3=0,$$

that is $x=0$, or $x=\frac{1}{2}$, or $x=-\frac{3}{2}$.

Thus the roots of the equation are 0, $\frac{1}{2}$, $-\frac{3}{2}$.

141. Since all the terms of any equation can be transposed to one side, an equation can always be written with all its terms on one side of the sign of equality, and zero on the other side.

It therefore follows from the last article that **the problem of solving an equation of any degree is the same as the problem of finding the factors of an expression of the same degree.**

Hence the process of solving *any* equation which involves only one unknown quantity is as follows:—

First write the equation with all its terms on one side of the sign of equality, and zero on the other side; then resolve the whole expression into factors, and the values obtained by equating each of these factors separately to zero will be the required roots.

In the following examples the resolution into factors can be performed by inspection.

Ex. 1. Solve the equation $x^2-3x=0$.

Since $x^2-3x=x(x-3)$,

we have $x(x-3)=0$.

Whence $x=0$, or $x-3=0$;

the roots required are therefore 0 and 3.

Ex. 2. Solve the equation $x^2-9=0$.

Since $x^2-9=(x-3)(x+3)$,

we have $(x-3)(x+3)=0$.

Whence $x-3=0$, or $x+3=0$;

the roots required are therefore 3 and -3.

Ex. 3. Solve the equation $x^2-2=0$.

Since $x^2-2=(x-\sqrt{2})(x+\sqrt{2})$,

we have $(x-\sqrt{2})(x+\sqrt{2})=0$.

Hence $x-\sqrt{2}=0$, or $x+\sqrt{2}=0$.

Thus $\sqrt{2}$ and $-\sqrt{2}$ are the required roots of the equation.

Ex. 4. Solve the equation $x^3-4x=0$.

Since $x^3-4x=x(x^2-4)=x(x-2)(x+2)$,

we have $x(x-2)(x+2)=0$.

Hence $x=0$, or $x-2=0$, or $x+2=0$.

Thus 0, 2 and -2 are the roots of the equation.

Ex. 5. Solve $9x^3=4x$.

Transposing, we have

$$9x^3-4x=0,$$

that is $x(9x^2-4)=0$,

or $x(3x-2)(3x+2)=0$.

Hence $x=0$, or $3x-2=0$, or $3x+2=0$,

that is $x=0$, or $x=\frac{2}{3}$, or $x=-\frac{2}{3}$.

Ex. 6. Solve $x^2+6=5x$.

Transposing, we have

$$x^2-5x+6=0,$$

that is $(x-2)(x-3)=0$.

Hence $x-2=0$, or $x-3=0$.

Thus 2 and 3 are the required roots.

EXAMPLES. XLIII.

Solve the equations:

1. $(x-1)(x+2)=0$.
2. $(x-3)(x-4)=0$.
3. $(x+1)(x+2)=0$.
4. $(2x+1)(2x-1)=0$.
5. $(3x-1)(3x+1)=0$.
6. $(2x-5)(x-4)=0$.
7. $x(x-2)(x+3)=0$.
8. $x(x+2)(x-4)=0$.
9. $x(x-3)(x+4)=0$.
10. $x(2x-1)(3x+4)=0$.
11. $x(5x-2)(6x-7)=0$.
12. $x(x-3)(3x+7)=0$.
13. $(x-1)(x-2)(x-3)(x-5)=0$.
14. $(x-2)(x-1)(x+1)(x+2)=0$.
15. $(3x-1)(4x+1)(5x-2)(2x+7)=0$.
16. $(2x-3)(3x-4)(4x-5)(5x+6)=0$.
17. $x^2-x=0$.
18. $x^2-2x=0$.
19. $x^2+3x=0$.
20. $2x^2-3x=0$.
21. $2x^2-5x=0$.
22. $3x^2+x=0$.
23. $x^2=5x$.
24. $2x^2=x$.
25. $3x^2=5x$.

26. $5x^2=-6x$. **27.** $x^2=bx$. **28.** $ax^2=bx$.
29. $6x^2=x^2+5$. **30.** $5(x^2+5)=3(x^2+25)$.
31. $5(x^2+4)=4(x^2+9)$. **32.** $2(x^2+7)=7(x^2+2)$.
33. $5(x^2+3)-(x-5)(x+5)=76$.
34. $7(x^2-1)-(x+3)(x-3)=56$.
35. $3x^2+(5x+2)^2=20x+32$. **36.** $17+3x=\frac{1}{2}(x+3)^2-28$.
37. $x^2-5x+6=0$. **38.** $x^2-7x+12=0$.
39. $x^2-12x+20=0$. **40.** $x^2-9x+20=0$.
41. $x^2-11x+28=0$. **42.** $x^2-25x+150=0$.
43. $x^2-5x-84=0$. **44.** $x^2-x-156=0$.
45. $x^2+5x-150=0$. **46.** $x^2+2x=3$. **47.** $x^2+4x=45$.
48. $x^2-3x=10$. **49.** $x^3+11x^2-180x=0$.
50. $x^3-x^2=132x$.

142. **A quadratic equation** is an equation which contains the unknown quantity to the second, but to no higher power. Thus $x^2=4$, $3x^2+4x=7$, and $ax^2+bx+c=0$ are quadratic equations.

When all the terms are transposed to one side, a quadratic equation must be of the form

$$ax^2+bx+c=0,$$

where a, b, c are supposed to represent known quantities.

We proceed to shew how to solve any quadratic equation.

As we have already remarked, the whole difficulty consists in finding the factors of the expression on the left side of the equation; and we have shewn in Art. 104 how this can be done by *changing the expression into an equivalent one which is the difference of two squares.*

We first apply the method to some examples; and it would be well to read Art. 104 again before considering these examples.

Ex. 1. Solve $x^2+12x+35=0$.

Since x^2+12x is made a perfect square by the addition of the *square of half the coefficient of* x, that is the square of 6; we add and subtract 6^2. The equation then becomes

$$x^2+12x+6^2-6^2+35=0,$$

that is $$(x+6)^2-1=0.$$

$$\therefore\ \{(x+6)+1\}\{(x+6)-1\}=0,$$

that is $$(x+7)(x+5)=0.$$

Hence $$x+7=0, \text{ or } x+5=0.$$

Thus $$x=-7, \text{ or } x=-5.$$

Ex. 2. Solve $3x^2=10x-3$.

By transposition we have

$$3x^2-10x+3=0.$$

We must first divide by 3 so as to make the coefficient of x^2 unity; the equation then becomes

$$x^2-\tfrac{10}{3}x+1=0,$$

Half the coefficient of x is $-\frac{5}{3}$, and therefore $x^2-\frac{10}{3}x$ will be made a perfect square by the addition of $(-\frac{5}{3})^2$; and adding and subtracting $(-\frac{5}{3})^2$, that is $\frac{25}{9}$, the equation becomes

$$x^2-\tfrac{10}{3}x+\tfrac{25}{9}-\tfrac{25}{9}+1=0;$$

$$\therefore\ (x-\tfrac{5}{3})^2-\tfrac{16}{9}=0,$$

that is $$(x-\tfrac{5}{3})^2-(\tfrac{4}{3})^2=0;$$

$$\therefore\ \{x-\tfrac{5}{3}+\tfrac{4}{3}\}\{x-\tfrac{5}{3}-\tfrac{4}{3}\}=0;$$

$$\therefore\ (x-\tfrac{1}{3})(x-3)=0.$$

Hence $$x-\tfrac{1}{3}=0, \text{ or } x-3=0.$$

Thus $$x=\tfrac{1}{3}, \text{ or } x=3.$$

Note. In most cases the factors can be written down at once, as in Arts. 99 and 100, without completing the square. The student should always see if this can be done: he will thus save himself a great deal of unnecessary work.

Ex. 3. Solve $4x-x^2=2$.

By transposition, we have

$$4x-x^2-2=0.$$

Change all the signs so as to make the coefficient of x^2 unity; then we have

$$x^2-4x+2=0.$$

Add and subtract 4, the square of half the coefficient of x. Then

$$x^2-4x+4-4+2=0;$$

$$\therefore\ (x-2)^2-2=0;$$

that is $$(x-2)^2-(\sqrt{2})^2=0;$$

$$\therefore\ (x-2+\sqrt{2})(x-2-\sqrt{2})=0.$$

Hence $$x-2+\sqrt{2}=0, \text{ or } x-2-\sqrt{2}=0.$$

Thus $$x=2-\sqrt{2}, \text{ or } x=2+\sqrt{2}.$$

EXAMPLES. XLIV.

Solve the following equations :—

1. $x^2-4x=45.$
2. $x^2-6x=91.$
3. $(x-3)^2=x+3.$
4. $(x-4)^2=x-2.$
5. $(x-1)(x-2)=20.$
6. $(x+1)(x+3)=2(x+3).$
7. $4x+3=x(x+2).$
8. $4x-3=x(2-x).$
9. $(x+1)^3=(x-1)^3+26.$
10. $(x-1)^3=(x+1)^3-56.$ 10 to 20
11. $9x^2+6x+1=0.$
12. $9x^2+16=24x.$
13. $x^2+150=53x.$
14. $x^2=2x+99.$
15. $3x^2+3=10x.$
16. $3x^2+11x=20.$
17. $4x^2+21x=18.$
18. $6x^2+55x=50.$
19. $24x^2=30x+75.$
20. $x^2-200=35x.$
21. $19x^2-39x+2=0.$
22. $17x^2+8=70x.$
23. $51x^2+19x=10.$
24. $110x^2-21x+1=0.$
25. $21x^2-13x=20.$
26. $21x^2+23x=20.$
27. $6x^2+6=13x.$
28. $6x^2=5x+1.$
29. $9x^2-63x+68=0.$
30. $16x^2+3=16x.$
31. $29x^2-41x-138=0.$
32. $29x^2+11x=138.$
33. $x^2-16=215-10x.$
34. $(x+2)^2=4(x-1)^2.$
35. $(x+6)^2=16(x-6)^2.$
36. $(x+8)^2=9x^2.$
37. $(x-7)^2=49(x+2)^2.$
38. $x^2-3ax+2a^2=0.$
39. $x^2+3a^2=4ax.$
40. $x^2+2ab=b^2+2ax.$
41. $4x^2+4ax=b^2-a^2.$
42. $x^2+2ax=b^2+2ab.$

143. To solve the equation

$$ax^2+bx+c=0.$$

Divide by a, the coefficient of x^2; the equation then becomes

$$x^2+\frac{b}{a}x+\frac{c}{a}=0.$$

Now add and subtract *the square of half the coefficient of* x, that is the square of $\frac{1}{2}\frac{b}{a}$. Then we have

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}=0.$$

The first three terms are now a perfect square, namely

$$\left(x+\frac{b}{2a}\right)^2.$$

Hence we have $$\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)=0,$$

that is $$\left(x+\frac{b}{2a}\right)^2-\left\{\sqrt{\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)}\right\}^2=0.$$

Hence [Art. 97]

$$\left\{x+\frac{b}{2a}-\sqrt{\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)}\right\}\left\{x+\frac{b}{2a}+\sqrt{\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)}\right\}=0.$$

We therefore have $$x+\frac{b}{2a}-\sqrt{\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)}=0,$$

or else $$x+\frac{b}{2a}+\sqrt{\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)}=0.$$

That is $$x=-\frac{b}{2a}+\sqrt{\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)},$$

or $$x=-\frac{b}{2a}-\sqrt{\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)}.$$

It follows from the above that a *quadratic equation has two and only two roots.*

144. Instead of working out every example from the beginning, we may use the general formula found in Art. 143, and substitute for a, b and c their values in the particular equation we are considering.

Thus, the roots of the equation $ax^2+bx+c=0$ being

$$-\frac{b}{2a}\pm\sqrt{\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)},$$

we find the roots of $3x^2-10x+3=0$ by putting 3 for a, -10 for b and 3 for c in the above formula.

Hence the roots of $3x^2-10x+3=0$ are

$$-\frac{-10}{6}\pm\sqrt{\left(\frac{100}{36}-\frac{3}{3}\right)},$$

that is $\frac{10}{6}\pm\frac{8}{6}$; and hence the roots are 3 and $\frac{1}{3}$.

145. In Art. 143 we found that the quadratic equation $ax^2 + bx + c = 0$ had *two* roots, namely,

$$-\frac{b}{2a} + \sqrt{\left(\frac{b^2}{4a^2} - \frac{c}{a}\right)} \text{ and } -\frac{b}{2a} - \sqrt{\left(\frac{b^2}{4a^2} - \frac{c}{a}\right)}.$$

If $\frac{b^2}{4a^2} - \frac{c}{a} = 0$, both the roots reduce to $-\frac{b}{2a}$, and are thus equal to one another. In this case we do not say that the equation has only *one* root, but that it has *two equal* roots.

It is clear that the roots are unequal unless $\frac{b^2}{4a^2} - \frac{c}{a}$ is zero.

Hence *in order that the roots of* $ax^2 + bx + c = 0$ *may be equal to one another, it is necessary and sufficient that* $\frac{b^2}{4a^2} - \frac{c}{a} = 0$, *that is, that* $b^2 - 4ac = 0$.

When $b^2 - 4ac = 0$, the expression $ax^2 + bx + c$ is a perfect square *so far as x is concerned;* for $ax^2 + bx + c$, that is, $a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ will then be $a\left(x + \frac{b}{2a}\right)^2$, since $\frac{b^2}{4a^2} = \frac{c}{a}$.

146. From the formula of Art. 143 it is clear that the roots of the equation $ax^2 + bx + c = 0$ are *irrational* unless $\frac{b^2}{4a^2} - \frac{c}{a}$ is a perfect square.

It is also clear that if either of the roots of a quadratic equation is irrational they are *both* irrational.

147. If $\frac{b^2}{4a^2} - \frac{c}{a}$ is negative, $\sqrt{\left(\frac{b^2}{4a^2} - \frac{c}{a}\right)}$ is *imaginary*— Sq. r a ne [Art. 106].

Hence the roots of $ax^2 + bx + c = 0$ are imaginary if $\frac{b^2}{4a^2} - \frac{c}{a}$ is negative, or if $b^2 - 4ac$ is negative.

It should be noticed that if either of the roots of a quadratic equation is imaginary, they are *both* imaginary.

148. By multiplying both sides of an equation by the same quantity, we do not destroy the equality, and therefore the new equation is satisfied by all the values which satisfy the original equation; if, however, we multiply by any integral expression *which contains the unknown quantity*, the new equation will be satisfied not only by the values which satisfy the original equation, but *also* by any value which makes the expression by which we multiply vanish.

For example, the equation $x^2=9$ is satisfied by the values $x=3$, or $x=-3$. If we multiply both sides of the equation by $x-2$, we have the new equation

$$x^2(x-2)=9(x-2);$$

and this new equation is satisfied not only by $x=3$, and by $x=-3$, but also by $x=2$.

Thus *additional roots* are introduced whenever both sides of an integral equation are *multiplied* by any integral expression which contains the unknown quantity.

When an equation contains fractions in whose denominators the unknown quantity occurs, the equation may, however, be multiplied by the L. C. M. of the denominators without introducing any additional roots; for we cannot divide both sides of the resulting equation by any one of the factors of the L. C. M. without re-introducing fractions.

The student must, however, be careful to multiply by the *lowest* common multiple of the denominators, for otherwise the resulting equations will have roots which are not roots of the original equation.

We may, for example, multiply both sides of the equation

$$\frac{2x}{x-1}-\frac{10}{x^2-1}=\frac{7}{x+1}$$

by x^2-1, the L.C.M. of the denominators of the fractions: we thus obtain

$$2x(x+1)-10=7(x-1),$$

and this new equation is not satisfied by either of the values obtained by equating to zero the factor by which we have multiplied, and hence no additional roots have been introduced by the multiplication.

If, however, we multiply the original equation by $(x-1)(x^2-1)$, we shall have

$$2x(x+1)(x-1)-10(x-1)=7(x-1)^2,$$

and one of the roots of this equation, namely $x=1$, is not a root of the original equation.

NOTE. The student should always remember that when both sides of an equation are divided by any factor *which contains the unknown quantity*, the resulting equation will not give *all* the roots of the original equation; to find the remaining roots we must equate to zero the factor by which we have divided.

For example, if we divide both sides of the equation

$$(x^2-4)(x+2)=(x^2-4)(2x+1)$$

by x^2-4, we obtain the equation

$$x+2=2x-1,$$

from which we obtain $x=3$. But $x=3$ is not the *only* solution of the original equation; to obtain the other solutions we must equate x^2-4 to zero.

EXAMPLES. XLV.

Solve the following equations:

1. $x+\frac{1}{x}=\frac{5}{2}.$

2. $x+\frac{1}{x}=\frac{13}{6}.$

3. $\frac{x}{2}+\frac{2}{x}=\frac{10}{3}.$

4. $\frac{x+5}{4}-\frac{4}{x+5}=\frac{3}{2}.$

5. $\frac{x}{x-1}+\frac{x-1}{x}=\frac{13}{6}.$

6. $\frac{x+1}{x+2}+\frac{x+2}{x+1}=\frac{29}{10}.$

7. $\frac{1}{x-2}-\frac{2}{x+2}=\frac{3}{5}.$ 8. $\frac{7}{x+4}+\frac{1}{4-x}=\frac{5}{3}.$

9. $\frac{x}{x-2}+\frac{4}{x+1}=3.$ 10. $\frac{5}{x+1}-\frac{10}{x+10}=\frac{2}{3x-3}.$

11. $\frac{1}{2}+\frac{1}{3+x}+\frac{1}{2+3x}=0.$ 12. $\frac{1}{3}+\frac{1}{x+3}+\frac{1}{x+11}=0.$

13. $3\frac{x-1}{x+1}-2\frac{x+1}{x-1}=5.$ 14. $\frac{3}{x-1}+\frac{4}{x-3}=\frac{15}{x+3}.$

15. $\frac{4x+3}{9}=\frac{6x-13}{18}-\frac{7x-46}{5x+3}.$ 16. $\frac{2x-3}{4}=\frac{x-5}{12}+\frac{5x-16}{x-1}.$

17. $\frac{2x-1}{2x+1}+\frac{13}{11}=\frac{3x+5}{3x-5}.$ 18. $\frac{2x-2}{2x-3}+\frac{3-3x}{3x-2}=\frac{5}{8x-12}.$

19. $\frac{3x}{x-2}-\frac{4}{x+3}+\frac{4}{2-x}=0.$ 20. $\frac{5x}{x-3}-\frac{6}{x+2}+\frac{19}{3-x}=0.$

21. $\frac{x-1}{x+1}+\frac{x+1}{x-1}=\frac{5x}{x^2-1}.$ 22. $\frac{1}{x^2-3x}+\frac{1}{x^2+4x}=\frac{9}{2x^2}.$

23. $\frac{1}{x^2-3x}-\frac{1}{9-x^2}=\frac{13}{16x}.$ 24. $\frac{1}{3x-6}+\frac{5}{4-x^2}+\frac{7x}{72\,(x+2)}=0.$

25. $\frac{1}{x^2-1}-\frac{1}{1-x}=\frac{7}{8}-\frac{1}{x+1}.$ 26. $\frac{1}{x^2-4}-\frac{3}{2-x}=1+\frac{1}{3x+6}.$

27. $\frac{1}{x}+\frac{1}{x+4}=\frac{1}{x+1}+\frac{1}{x+2}.$ 28. $\frac{x}{x+1}+\frac{x+1}{x+2}=\frac{x-2}{x-1}+\frac{x-1}{x}.$

29. $\frac{x}{x-3}-\frac{x-3}{x}+\frac{x}{x+3}-\frac{x+3}{x}=\frac{2}{3}.$

30. $\frac{x+3}{x+1}+\frac{x-6}{x-4}=\frac{x+4}{x+2}+\frac{x-5}{x-3}.$

31. $\frac{x-3}{x-4}-\frac{x-4}{x-5}=\frac{x-6}{x-7}-\frac{x-7}{x-8}.$

32. $x^2+2a^2=3ax.$ 33. $9x^2-6ax=a^2-b^2.$

34. $a\,(x^2-1)+x\,(a^2-1)=0.$ 35. $a\,(x^2+1)=x\,(a^2+1).$

36. $x^2-2\,(a-b)\,x+b^2=2ab.$ 37. $x^2+2\,(b-c)\,x+c^2=2bc.$

38. $(b-c)\,x^2+(c-a)\,x+(a-b)=0.$

39. $(a+b)\,x^2+cx-a-b-c=0.$ 40. $abx^2-(a^2+b^2)\,x+ab=0.$

41. $bcx^2+(b^2+c^2)\,x+bc=0.$ 42. $(a^2-b^2)\,(x^2-1)=4abx.$

43. $(b^2-a^2)\,(x^2+1)=2\,(a^2+b^2)\,x.$

44. $x+\frac{1}{a}=a+\frac{1}{x}$.

45. $\frac{a}{a+x}+\frac{a}{a-x}=4$.

46. $\frac{1}{x-a}+\frac{1}{x-b}=\frac{1}{a}+\frac{1}{b}$.

47. $\frac{a}{x-a}+\frac{b}{x-b}=\frac{a}{b}+\frac{b}{a}$.

48. $\frac{a^2}{x-b}+\frac{b^2}{x-a}=a+b$.

49. $\frac{x}{a}+\frac{a}{x}=\frac{b}{a}+\frac{a}{b}$.

50. $\frac{1}{x+a+b}=\frac{1}{x}+\frac{1}{a}+\frac{1}{b}$.

51. $\frac{1}{x-a-b}=\frac{1}{x}-\frac{1}{a}-\frac{1}{b}$.

52. $\frac{1}{x+a}+\frac{1}{x+b}=\frac{1}{c+a}+\frac{1}{c+b}$.

53. $\frac{x}{x+a}+\frac{x}{x+b}=\frac{c}{c+a}+\frac{c}{c+b}$.

54. $\frac{x+a}{x-a}+\frac{x+b}{x-b}=\frac{x-a}{x+a}+\frac{x-b}{x+b}$.

55. $\frac{a}{x+a}+\frac{b}{x+b}=\frac{a-c}{x+a-c}+\frac{b+c}{x+b+c}$.

149. **Irrational Equations.** An *irrational* equation is one in which square or other roots of expressions containing the unknown quantity occur.

In order to rationalize an equation it is first written with one of the irrational terms standing by itself on one side of the sign of equality: both sides are then raised to the lowest power necessary to rationalize the isolated term; and the process is repeated as often as may be necessary.

Ex. 1. Solve the equation $\sqrt{(x^2-9)}+x=9$.

By transposition we have

$$\sqrt{(x^2-9)}=9-x.$$

Square both sides; then

$$x^2-9=(9-x)^2;$$

$$\therefore\quad 18x=90;\ \therefore\ x=5.$$

Ex. 2. Solve the equation

$$\sqrt{(2x+8)}-2\sqrt{(x+5)}=2.$$

Squaring both sides, we have

$$2x+8+4(x+5)-4\sqrt{(2x+8)}\sqrt{(x+5)}=4,$$

that is $$6x+24-4\sqrt{(2x+8)}\sqrt{(x+5)}=0;$$

$$\therefore\quad 3x+12=2\sqrt{(2x+8)}\sqrt{(x+5)}.$$

Square both sides of the last equation, and we have

$$9x^2+72x+144=4(2x+8)(x+5);$$

$$\therefore\quad x^2-16=0.$$

Hence $$x=4 \text{ or } x=-4.$$

NOTE. If we put 4 for x in the original equation, we should have

$$\sqrt{16}-2\sqrt{9}=2, \text{ that is } 4-6=2,$$

which is not true.

Again, if we put -4 for x, we should have

$$\sqrt{0}-2\sqrt{1}=2,$$

which is not true.

Thus neither of the values we found for x *appears* to satisfy the equation.

The failure is however only apparent; and is due to our having supposed that $\sqrt{(2x+8)}$ and $\sqrt{(x+5)}$ were necessarily *positive*, whereas every algebraical quantity has *two* square roots, one positive and the other negative, and there is no symbol which represents one only to the exclusion of the other.

Bearing this in mind we shall find that $x=4$ does satisfy the equation, for the condition is

$$\pm\sqrt{16}-2(\pm\sqrt{9})=2, \text{ that is } \pm 4-(\pm 6)=2,$$

which is true when the lower signs are taken.

If an irrational equation is to be solved on the supposition that the radicals it contains are to be necessarily positive (or necessarily negative) the equation must be accompanied by a definite statement to that effect.

Ex. 3. Solve the equation

$$\sqrt{(2x+7)}+\sqrt{(3x-18)}+\sqrt{(7x+1)}=0.$$

By transposition, we have

$$\sqrt{(2x+7)}+\sqrt{(3x-18)}=-\sqrt{(7x+1)}.$$

Square both members; then

$$2x+7+2\sqrt{(2x+7)}\sqrt{(3x-18)}+3x-18=7x+1.$$

By transposition and division by 2, we have

$$\sqrt{(2x+7)}\sqrt{(3x-18)}=x+6.$$

Square both members; then

$$(2x+7)(3x-18)=(x+6)^2;$$

$$\therefore \quad 5x^2-27x-162=0.$$

Hence $\quad x=9$ or $x=-\frac{18}{5}$.

EXAMPLES. XLVI.

1. $\sqrt{(x-3)}=3.$
2. $\sqrt{(x-7)}=4.$
3. $\sqrt{(3x+1)}=4.$
4. $5\sqrt{(x+7)}=4\sqrt{(3x-2)}.$
5. $3\sqrt{(x+3)}=2\sqrt{(3x+6)}.$
6. $\sqrt{(x+5)}=2\sqrt{(3x+4)}.$
7. $3\sqrt{(x+7)}=5\sqrt{(3x-2)}.$
8. $\sqrt{(x+2)}=x.$
9. $\sqrt{(x+20)}=x.$
10. $x+\sqrt{(x+1)}=5.$
11. $x-\sqrt{(x+2)}=4.$
12. $x-2+3\sqrt{(x-2)}=0.$
13. $x-5+2\sqrt{(x-5)}=0.$
14. $\sqrt{(9+4x)}=2x-3.$
15. $3x=5+\sqrt{(30x-71)}.$
16. $2x-5\sqrt{x}=3.$
17. $x+3+\sqrt{(x+3)}=6.$
18. $2x+1=\sqrt{(6x+3)}.$
19. $7x=\sqrt{(3x-11)}+33.$
20. $\sqrt{(x+10)}+\sqrt{(x+1)}=1.$
21. $\sqrt{x}+\sqrt{(5x+1)}=5.$
22. $\sqrt{x}+\sqrt{(5x+1)}=2.$
23. $\sqrt{(2x+9)}-\sqrt{(x+4)}=1.$
24. $\sqrt{(7x+1)}-\sqrt{(3x+10)}=1.$
25. $\sqrt{(2x+11)}-\sqrt{(2x-5)}=2.$
26. $\sqrt{(4x+1)}-\sqrt{(x+3)}=2.$
27. $\sqrt{(8x+5)}-2\sqrt{(2x+1)}=1.$
28. $\sqrt{(x+4)}+\sqrt{(x+20)}-2\sqrt{(x+11)}=0.$
29. $\sqrt{(4x+1)}-\sqrt{(x+3)}=\sqrt{(x-2)}.$
30. $\sqrt{(2x+4)}+\sqrt{(3x+7)}=\sqrt{(12x+9)}.$
31. $\sqrt{(6x+1)}+\sqrt{2(1-x)}=\sqrt{(7x+6)}.$
32. $2\sqrt{(2x+3)}-\sqrt{(5x+1)}=\sqrt{2(x-1)}.$
33. $\sqrt{(x-1)(x-2)}+\sqrt{(x-3)(x-4)}=\sqrt{2}.$
34. $\sqrt{(x+1)}+\dfrac{1}{\sqrt{(x+1)}}=2.$
35. $\sqrt{(x+1)}+\sqrt{x}=\dfrac{3}{\sqrt{(1+x)}}.$
36. $\sqrt{(3+x)}+\sqrt{x}=\dfrac{5}{\sqrt{x}}.$

37. $\sqrt{\left(6x+\frac{3}{2}\right)}=\sqrt{(x+4)}+\frac{3}{\sqrt{(x+4)}}$.

38. $\sqrt{(a-x)}+\sqrt{(b-x)}=\sqrt{(a+b-2x)}$.

39. $\sqrt{(ax+b^2)}+\sqrt{(bx+a^2)}=a-b$.

40. $\sqrt{(a+x)}+\sqrt{(b+x)}=\sqrt{(a+b+2x)}$.

41. $\sqrt{(a-x)}+\sqrt{(b-x)}=\sqrt{(2a+2b)}$.

150. **Relations between the roots and the coefficients of a quadratic equation.**

We have seen [Art. 104] that x^2+px+q can always be expressed in the form $(x-\alpha)(x-\beta)$.

We have also seen [Art. 139] that if

$$x^2+px+q=(x-\alpha)(x-\beta),$$

then α and β are the roots of the equation

$$x^2+px+q=0.$$

But, if

$$x^2+px+q=(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta,$$

we have $\alpha+\beta=-p$ and $\alpha\beta=q$.

Hence in the equation

$$x^2+px+q=0,$$

the sum of the roots is $-p$ and *the product of the roots is* q.

The equation $ax^2+bx+c=0$ becomes, on dividing by a,

$$x^2+\frac{b}{a}x+\frac{c}{a}=0.$$

Hence, from the above, if α and β be the roots of

$$ax^2+bx+c=0,$$

we have $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

The above relations between the roots of a quadratic equation and the coefficients of the different powers of the unknown quantity are of great importance.

Analogous relations hold good for equations of the third and of higher degrees. [See *Treatise on Algebra*, Art. 129.]

Or thus.

We have found in Art. 143 that the roots of

$$ax^2 + bx + c = 0$$

are
$$-\frac{b}{2a} \pm \sqrt{\left\{\frac{b^2}{4a^2} - \frac{c}{a}\right\}}.$$

$x = -b \pm \sqrt{b^2 - 4}$ [illegible]

Call these roots α and β respectively;

then
$$\alpha = -\frac{b}{2a} + \sqrt{\left\{\frac{b^2}{4a^2} - \frac{c}{a}\right\}},$$

and
$$\beta = -\frac{b}{2a} - \sqrt{\left\{\frac{b^2}{4a^2} - \frac{c}{a}\right\}}.$$

By addition, we have

$$\alpha + \beta = -\frac{b}{a}.$$

By multiplication, we have

$$\alpha\beta = \left(-\frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2} - \frac{c}{a}\right) = \frac{c}{a}.$$

151. Although we cannot in all cases find the roots of a *given equation*, it is very easy to solve the converse problem, namely, the problem of finding an equation which has *given roots*.

For example, to find the equation whose roots are 4 and 5.

We want to find an equation which is satisfied when $x = 4$, or when $x = 5$; that is when $x - 4 = 0$, or when $x - 5 = 0$; and in no other case. The equation required must be

$$(x - 4)(x - 5) = 0;$$

for this is an equation which is a true statement when $x - 4 = 0$, or when $x - 5 = 0$, and in no other case.

If we get rid of the brackets by multiplying out, the equation becomes

$$x^2-9x+20=0\,^*.$$

Again, to find the equation whose roots are 2, 3 and -4.

We have to find an equation which is satisfied when $x-2=0$, or when $x-3=0$, or when $x+4=0$, and in no other case. The equation must therefore be

$$(x-2)(x-3)(x+4)=0,$$

that is $$x^3-x^2-14x+24=0.$$

Similarly, the equation whose roots are 0, -1, and $-\frac{1}{2}$ is

$$x(x+1)(x+\tfrac{1}{2})=0,$$

that is $$x^3+\tfrac{3}{2}x^2+\tfrac{1}{2}x=0.$$

Ex. 1. Find the equation whose roots are the squares of the roots of the equation $x^2+5x-7=0$.

Let α, β be the roots of the given equation; then α^2, β^2 will be the roots of the required equation. Hence the required equation is

$$(x-\alpha^2)(x-\beta^2)=0,$$

that is $$x^2-(\alpha^2+\beta^2)\,x+\alpha^2\beta^2=0 \text{ ....(i).}$$

We have therefore to find $\alpha^2+\beta^2$ and $\alpha^2\beta^2$. Now, by Art. 150, we have

$$\alpha+\beta=-5,$$

and $$\alpha\beta=-7.$$

Hence $$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=(-5)^2-2(-7)$$
$$=25+14=39;$$

also $$\alpha^2\beta^2=49.$$

Substituting in (i), we have the required equation, namely

$$x^2-39x+49=0.$$

---

* The equation $x^2-9x+20=0$ is certainly *an* equation with the proposed roots; but to prove that it is the *only* equation with the proposed and with no other roots we must assume that *every equation has a root.*

If the equation $x^5+7x^2-2=0$, for example, had no roots, then $(x-4)(x-5)(x^5+7x^2-2)=0$ would be an equation with the proposed roots and with no others.

The proof of the proposition that every equation has a root is too difficult for an elementary book.

NOTE. We might have obtained the required equation by finding the roots of the given equation, and squaring those roots; but it is best to use the relations found in Art. 150.

Ex. 2. If α, β are the roots of the equation

$$ax^2+bx+c=0,$$

find the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

The required equation is

$$\left(x-\frac{\alpha}{\beta}\right)\left(x-\frac{\beta}{\alpha}\right)=0,$$

that is $$x^2-\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)x+1=0.$$

Now $$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta};$$

and, by Art. 150, we have

$$\alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}.$$

Hence $$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{b^2}{a^2}-2\frac{c}{a};$$

$$\therefore\ \frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{\frac{b^2}{a^2}-\frac{2c}{a}}{\frac{c}{a}}=\frac{b^2}{ac}-2.$$

Hence the required equation is

$$x^2-\left(\frac{b^2}{ac}-2\right)x+1=0,$$

or, multiplying by ac,

$$acx^2-(b^2-2ac)x+ac=0.$$

Ex. 3. Shew that no numerical value of x will make x^2-4x+5 zero; and find its least value.

Since $$x^2-4x+5=(x-2)^2+1,$$

and $(x-2)^2$ is always positive, $1+(x-2)^2$ is always greater than 1, except when $x-2=0$, and then it is equal to 1.

Thus x^2-4x+5 can never be zero, and its least value is 1.

EXAMPLES. XLVII.

Write down the equations whose roots are

1. 2 and -2.
2. 3 and -4.
3. -3 and -2.
4. $\frac{1}{2}$ and $-\frac{1}{2}$.
5. $\frac{1}{3}$ and $-\frac{1}{4}$.
6. $-\frac{1}{3}$ and $-\frac{1}{2}$.
7. 0 and 3.
8. 0 and -4.
9. $5, -3$ and 0.
10. $\sqrt{2}$ and $-\sqrt{2}$.
11. $\sqrt{5}$ and $-\sqrt{5}$.
12. $0, \sqrt{3}$ and $-\sqrt{3}$.
13. $2-\sqrt{3}$ and $2+\sqrt{3}$.
14. $5-\sqrt{7}$ and $5+\sqrt{7}$.
15. $a-\sqrt{b}$ and $a+\sqrt{b}$.

16. Write down the product of the roots of each of the following equations:

(i) $x^2+4x+1=0$, (ii) $x^2+7x-2=0$, (iii) $3x^2+5x-7=0$,
(iv) $5x^2-x-1=0$, (v) $9ax^2+3bx+4c=0$.

17. Write down the sum of the roots of each of the following equations:

(i) $x^2-4a^2=0$, (ii) $x^2+3x-5=0$, (iii) $x^2-5x+3=0$,
(iv) $2x^2-x-7=0$, (v) $6ax^2+7bx+8=0$.

18. Shew that, if α, β be the roots of $ax^2+bx+c=0$, then

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{b}{c}=0.$$

19. Shew that, if α, β be the roots of the equation $ax^2+bx+c=0$, then

(i) $\alpha^2+\beta^2=\dfrac{b^2-2ac}{a^2}$, (ii) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}=\dfrac{b^2-2ac}{ac}$,

and (iii) $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}=\dfrac{b^2-2ac}{c^2}$.

20. Find the sum of the squares of the roots of the equation

$$x^2+4x+2=0.$$

21. Find the sum of the squares of the roots of the equation

$$x^2+4px+p^2=0.$$

22. Shew that the sum of the squares of the roots of the equation $x^2+ax+b=0$ is the same as the sum of the squares of the roots of the equation $x^2+3ax+b+4a^2=0$.

23. For what value of a are the roots of the equation

$$3x^2+4x+a=0$$

equal to one another?

24. For what values of a are the roots of the equation

$$4x^2+(1+a)x+1=0$$

equal to one another?

25. Find the value of a in order that one of the roots of $100x^2+60x+a=0$ may be double the other.

26. Shew that one of the roots of $x^2+px+q=0$ is double the other, if $9q=2p^2$.

27. Shew that, if α, β are the roots of $2x^2-5x+3=0$, the equations whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ is $6x^2-13x+6=0$.

28. Shew that, if α, β be the roots of $2x^2-15x+4=0$, the equations whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ is $8x^2-209x+8=0$.

29. Form the equation whose roots are $\alpha+\beta$ and $\frac{1}{\alpha}+\frac{1}{\beta}$, where α, β are the roots of $x^2-11x+22=0$.

30. If α, β are the roots of $x^2+7x+9=0$, find the equation whose roots are $\frac{\alpha+\beta}{\alpha}$ and $\frac{\alpha+\beta}{\beta}$.

31. Shew that the roots of $x^2-(p^2-2q)x+q^2=0$ are the squares of the roots of $x^2+px+q=0$.

32. Find the condition that the roots of $a^2x^2+b^2x+c^2=0$ may be the squares of the roots of $ax^2+bx+c=0$.

33. Shew that, if α, β are the roots of $px^2+qx+r=0$, the equation whose roots are $\alpha+\beta$ and $\frac{\alpha\beta}{\alpha+\beta}$ is $pqx^2+(pr+q^2)x+qr=0$.

34. Shew that, if α, β are the roots of $ax^2+bx+c=0$, the equation whose roots are $\alpha\beta$ and $\frac{1}{\alpha\beta}$ is $acx^2-(a^2+c^2)x+ac=0$; shew also that the equation whose roots are $\alpha+\beta$ and $\frac{1}{\alpha+\beta}$ is

$$abx^2+(a^2+b^2)x+ab=0.$$

35. Shew that, if the difference between the roots of $x^2+ax+b=0$ is equal to the difference between the roots of $x^2+px+q=0$, then

$$a^2-4b=p^2-4q.$$

36. Shew that, if α, β are the roots of $x^2+px+q=0$, the equation whose roots are $(\alpha+\beta)^2$ and $(\alpha-\beta)^2$ is

$$x^2-2(p^2-2q)x+p^2(p^2-4q)=0.$$

37. If α, β are the roots of $x^2+px+q=0$, shew that the equation whose roots are $\alpha+\dfrac{1}{\beta}$ and $\beta+\dfrac{1}{\alpha}$ is $qx^2+p(1+q)x+(1+q)^2=0$.

38. Shew that, if α, β are the roots of $ax^2+bx+c=0$, the roots of the equation $(2b^2+ac)x^2+3abx+a^2=0$ are $\dfrac{1}{\alpha+2\beta}$ and $\dfrac{1}{\beta+2\alpha}$.

CHAPTER XV.

EQUATIONS OF HIGHER DEGREE THAN THE SECOND.

153. It is beyond the range of this book to shew how to solve equations of higher degree than the second, when the equations are in their most general forms; we give however some easy examples of such equations.

Ex. 1. Solve $x^4-6x^2+8=0$.

Here the equation contains only two powers of the unknown quantity one of which is the *square* of the other; we therefore proceed as in Art. 142.

We have $x^4-6x^2+9-9+8=0$,

that is $(x^2-3)^2-1=0$.

Hence $\{(x^2-3)-1\}\{(x^2-3)+1\}=0$;

$\therefore \quad x^2-4=0$, giving $x=\pm 2$,

or $x^2-2=0$, giving $x=\pm\sqrt{2}$.

Thus there are four roots,

$$\pm 2, \quad \pm\sqrt{2}.$$

Ex. 2. Solve $(x^2+x)^2+4(x^2+x)-12=0$.

In this, as in the former example, the unknown quantity occurs in two terms, one of which is the *square* of the other. In all such cases we can bring all the terms over to the left side of the sign of equality, and then resolve the expression on the left into two factors as in the preceding chapter.

In the present instance the factors can be seen by inspection.

For $A^2+4A-12=(A+6)(A-2)$;

therefore the equation may be written

$$\{(x^2+x)+6\}\{(x^2+x)-2\}=0.$$

Hence $x^2+x+6=0$, or $x^2+x-2=0$.

The roots of

$$x^2+x+6=0 \text{ are } -\frac{1}{2}\pm\frac{\sqrt{-23}}{2}.$$

The roots of

$$x^2+x-2=0 \text{ are } 1 \text{ and } -2.$$

Hence the given equation has the four roots

$$1, \quad -2, \quad -\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-23}.$$

Ex. 3. Solve

$$\frac{x^2}{x+2}+\frac{x+2}{x^2}=\frac{37}{6}.$$

Here one of the terms in which x occurs is the reciprocal of the other. If we put $y=\dfrac{x^2}{x+2}$, we find y from the quadratic equation

$$y+\frac{1}{y}=\frac{37}{6},$$

or, multiplying by $6y$,

$$6y^2-37y+6=0,$$

that is

$$(6y-1)(y-6)=0.$$

Hence $y=\dfrac{1}{6}$, or $y=6$.

Thus $\dfrac{x^2}{x+2}=\dfrac{1}{6}$, or $\dfrac{x^2}{x+2}=6$.

In the first case, we have $6x^2-x-2=0$, the roots of which are $\frac{2}{3}$ and $-\frac{1}{2}$.

In the second case, we have

$$x^2-6x-12=0,$$

the roots of which are $3\pm\sqrt{21}$.

Hence the given equation has the four roots,

$$\tfrac{2}{3}, \quad -\tfrac{1}{2}, \quad 3\pm\sqrt{21}.$$

Ex. 4. Solve $4x^2-6x+3\sqrt{(2x^2-3x+7)}=30$.

Equations of this form, in which the ratio of the coefficients of x^2 and x in the terms under the radical is equal to the ratio of the coefficients of x^2 and x in the terms outside the radical, are not uncommon.

To solve the equation, put

$$\sqrt{(2x^2-3x+7)}=y,$$

then $$2x^2-3x+7=y^2,$$

$$2x^2-3x=y^2-7,$$

and $$4x^2-6x=2y^2-14.$$

Hence, from the given equation, we have

$$2y^2-14+3y=30;$$

that is $$2y^2+3y-44=0,$$

that is $$(y-4)(2y+11)=0;$$

hence $$y=4, \quad \text{or} \quad y=-\tfrac{11}{2};$$

$$\therefore \quad y^2=16, \quad \text{or} \quad y^2=\tfrac{121}{4}.$$

Since $y^2=2x^2-3x+7$, we have

I. $$2x^2-3x+7=16,$$

that is $$2x^2-3x-9=0,$$

that is $$(2x+3)(x-3)=0;$$

$$\therefore \quad x=3, \quad \text{or} \quad x=-\tfrac{3}{2}.$$

II. $$2x^2-3x+7=\tfrac{121}{4},$$

that is $$2x^2-3x-\tfrac{93}{4}=0,$$

the roots of which are $\dfrac{3}{4}\pm\dfrac{\sqrt{195}}{4}$.

154. We shall prove in a subsequent chapter that if a rational and integral expression containing x vanishes when any particular value a is given to x, then $x-a$ is a factor of that expression.

For example, x^3-7x+6 vanishes when 2 is put for x; therefore, by the above theorem, $x-2$ is a factor, and by division we find that

$$x^3-7x+6=(x-2)(x^2+2x-3).$$

Again, x^3-4x^2+2x+1 vanishes when $x=1$; therefore $x-1$ is a factor, and by division we find that

$$x^3-4x^2+2x+1=(x-1)(x^2-3x-1).$$

155. It follows from the theorem enunciated in the last Article that when one root of an equation is known, the degree of the equation can be lowered.

Ex. 1. One root of the equation $x^3-7x+6=0$ is 2; find the other roots.

Since x^3-7x+6 vanishes when $x=2$, $x-2$ must be a factor, and by division we find that

$$x^3-7x+6=(x-2)(x^2+2x-3).$$

Hence $$(x-2)(x^2+2x-3)=0.$$

Hence the other roots of the equation are those given by

$$x^2+2x-3=0,$$

that is $$(x+3)(x-1)=0.$$

Thus the cubic equation $x^3-7x+6=0$ has the three roots 1, 2, and -3.

Ex. 2. Solve the equation $x^3-1=0$.

Since $$x^3-1=(x-1)(x^2+x+1),$$

we have $$(x-1)(x^2+x+1)=0.$$

Hence $$x=1;$$

or else $$x^2+x+1=0,$$

the roots of which are

$$-\tfrac{1}{2}\pm\sqrt{(-\tfrac{3}{4})}.$$

Hence there are three roots of the equation $x^3=1$, one being real and the other two imaginary. Thus there are three quantities whose cubes are equal to 1; that is, there are *three cube roots* of 1, which are

$$1,\ -\tfrac{1}{2}+\sqrt{(-\tfrac{3}{4})} \text{ and } -\tfrac{1}{2}-\sqrt{(-\tfrac{3}{4})}.$$

Ex. 3. For what values of x will

$$x^4-25x^2+30x-9,$$

and $$x^4-8x^3+19x^2-14x+3$$

both vanish?

If both expressions vanish for the same value of x, say $x=a$, they will both have $x-a$ as a factor. Now we can find the common factor of any two expressions by the ordinary process of finding their H.C.F.

In the present case the H.C.F. will be found to be

$$x^2-5x+3;$$

and since x^2-5x+3 is a factor of both expressions, and is their highest common factor, both expressions will vanish for the values of x given by

$$x^2-5x+3=0,$$

and for no other values. Thus the values required are the roots of

$$x^2-5x+3=0,$$

and these roots are

$$\frac{5\pm\sqrt{13}}{2}.$$

EXAMPLES. XLVIII.

Find the roots of the following equations:

1. $x^4-5x^2+4=0.$

2. $x^4-2x^2-8=0.$

3. $x^4-10x^2+9=0.$

4. $x^4-7x^2-18=0.$

5. $\dfrac{x^2-1}{9}+\dfrac{1}{x^2}-1=0.$

6. $x^2+\dfrac{100}{x^2}=29.$

7. $x^2+\dfrac{1}{x^2}=a^2+\dfrac{1}{a^2}.$

8. $x^2-\dfrac{1}{x^2}=a^2-\dfrac{1}{a^2}.$

9. $x^4+\dfrac{1}{x^4}=a^4+\dfrac{1}{a^4}.$

10. $(x^2-x)^2-8(x^2-x)+12=0.$

11. $(x^2+x)^2-22(x^2+x)+40=0.$

12. $(x^2+x)(x^2+x+1)=42.$

13. $\dfrac{x^2}{x+1}+\dfrac{x+1}{x^2}=2.$

14. $\dfrac{x}{x^2+1}+\dfrac{x^2+1}{x}=\dfrac{5}{2}.$

15. $\dfrac{x^2}{x-1}+\dfrac{x-1}{x^2}=\dfrac{17}{4}.$

16. $(x^2+x+1)\left(x^2+x-\dfrac{3}{2}\right)+1=0.$

17. $(x^2+x+1)(x^2+x+2)=12.$

18. $\left(x+\dfrac{1}{x}\right)^2+4\left(x+\dfrac{1}{x}\right)=12.$

19. $\dfrac{x^2+2}{x^2+4x+1}+\dfrac{x^2+4x+1}{x^2+2}=\dfrac{5}{2}.$

20. $x^2+x+1=\dfrac{42}{x^2+x}.$

21. $2x^2-4x-\sqrt{x^2-2x-3}=9.$

22. $x^2+\sqrt{x^2+3x+7}=23-3x.$ **23.** $2x^2+6x=1-\sqrt{(x^2+3x+1)}.$

24. $x^2+\sqrt{(4x^2+24x)}=24-6x.$

25. $2(2x-3)(x-4)-\sqrt{(2x^2-11x+15)}=60.$

26. $x^2+(x-2)(x-3)+\sqrt{2x^2-5x+6}=6.$

27. $(x+5)(x-2)-36=\sqrt{(x+4)(x-1)}.$

Solve the following cubic equations having given one of the roots of each.

28. $x^3-2x+1=0$, $[x=1]$. **29.** $x^3-5x+4=0$, $[x=1]$.

30. $x^3-49x+120=0$, $[x=3]$. **31.** $x^3-3x^2-7x+21=0$, $[x=3]$.

32. $x^3-2x^2-7x-4=0$, $[x=-1]$.

33. $5x^3-15x^2+3x+14=0$, $[x=2]$.

34. For what values of x will x^3+2x^2+9 and $x^3-4x+15$ both vanish?

35. The equations

$$\begin{aligned}2x^3+21x^2+34x-105&=0,\\ 2x^3-\quad x^2-76x-105&=0\end{aligned}$$

have one root in common; find it.

36. Solve the following equations:

(i) $x^3-21x+20=0$,
(ii) $x^3+2x^2-9x-18=0$,
(iii) $x^3-x^2-22x+40=0$,
(iv) $x^3-3x^2-60x-100=0$,
(v) $x^3-19x+30=0$,
(vi) $2x^3+7x^2-5x-4=0$,
(vii) $4x^3+4x^2-9x-9=0$.

CHAPTER XVI.

SIMULTANEOUS EQUATIONS OF THE SECOND DEGREE.

156. WE now proceed to consider simultaneous equations one at least of which is of the second degree.

We first take the case of two equations which contain two unknown quantities, one equation being of the first degree and the other of the second degree.

Example. Solve the equations

$$x+2y=5, \quad x^2+2y^2=9.$$

From the first equation we have

$$x=5-2y.$$

Substituting this value of x in the second equation, we have

$$(5-2y)^2+2y^2=9;$$

$$\therefore \quad 6y^2-20y+16=0;$$

$$\therefore \quad 3y^2-10y+8=0;$$

that is

$$(3y-4)(y-2)=0.$$

Hence

$$y=\frac{4}{3} \text{ or } y=2.$$

If $y=\frac{4}{3}$, $x=5-2y=5-\frac{8}{3}=\frac{7}{3}$;

and if $y=2$, $x=5-2y=5-4=1$.

Thus we have

$$x=\tfrac{7}{3},\ y=\tfrac{4}{3}; \text{ or } x=1,\ y=2.$$

[The result should not be written in the ambiguous form $x=\frac{7}{3}$ or 1, $y=\frac{4}{3}$ or 2.]

From the above example it will be seen that two equations, one of which is of the first degree and the other of the second degree, can be solved in the following manner :—

From the equation of the first degree find the value of one of the unknown quantities in terms of the other unknown quantity and the known quantities, and substitute this value in the equation of the second degree: one of the unknown quantities is thus eliminated, and a quadratic equation is obtained the roots of which are the values of the unknown quantity which is retained.

Ex. 1. Solve the equations

$$3x+4y=5, \quad 2x^2-xy+y^2=22.$$

From the first equation, $x=\frac{5-4y}{3}$. Hence, substituting in the second, we have

$$2\left(\frac{5-4y}{3}\right)^2-y\left(\frac{5-4y}{3}\right)+y^2=22,$$

which reduces to

$$53y^2-95y-148=0,$$

that is $$(y+1)(53y-148)=0.$$

Hence $$y=-1, \text{ or } y=\frac{148}{53}.$$

If $$y=-1, \; x=\frac{5-4y}{3}=3.$$

And, if $$y=\frac{148}{53}, \; x=\frac{5-4y}{3}=-\frac{109}{53}.$$

Ex. 2. Solve $xy+x=25, \quad 2xy-3y=28.$

Multiply the first equation by 2, and subtract the second ; then

$$2x+3y=50-28=22.$$

Hence $$y=\frac{22-2x}{3};$$

and substituting in the first equation, we have

$$x\left(\frac{22-2x}{3}\right)+x=25;$$

$$\therefore \qquad 2x^2-25x+75=0,$$

whence we have $x=5$ or $x=\frac{15}{2}$, the corresponding values of y being 4 and $\frac{7}{3}$ respectively.

Ex. 3. Solve

$$2x^2-3x-4y=47,$$
$$3x^2+4x+2y=89.$$

Multiply the second equation by 2; then

$$6x^2+8x+4y=178.$$

Adding the last equation to the first, we have

$$8x^2+5x=225,$$

whence $x=5$ or $x=-\frac{45}{8}$. The values of y are then found by substituting in either of the given equations.

EXAMPLES. XLIX.

Solve the following equations:

1. $x+y=6,$
$x^2-y^2=24.$

2. $x-y=10,$
$x^2+y^2=58.$

3. $3x+3y=10,$
$xy=1.$

4. $2x+3y=0,$
$4x^2+9xy+9y^2=72.$

5. $2x-5y=0,$
$x^2-3y^2=13.$

6. $x+y=15,$
$4x-4y=xy.$

7. $2x-y=5,$
$x+3y=2xy.$

8. $3x-31=5y,$
$x^2+5xy+25=y^2.$

9. $3x+2y=5,$
$x^2-4xy+5y^2=2.$

10. $2x-7y=25,$
$5x^2+4xy+3y^2=23.$

11. $x+y=2,$
$\frac{2}{x}+\frac{3}{y}=6.$

12. $x+2y=7,$
$\frac{3}{x}+\frac{6}{y}=5.$

13. $x+y=5,$
$1 \quad 5$

14. $x-y=1,$
$\frac{x}{y}-\frac{y}{x}=\frac{5}{6}.$

15. $2x-y=4,$
$\frac{3}{x}+\frac{4}{y}=1.$

17. $xy+x=15,$
$xy-x=8.$

18. $xy+2x=5,$
$2xy-y=3.$

19. $x^2-y=29,$
$x^2+x+y=49.$

20. x^2+3x
$2x^2-5x+3$

Ana

157. It should be remarked that we cannot solve *any* two equations which are both of the second degree; for the elimination of one of the unknown quantities will frequently lead to an equation of higher degree than the second, from which the remaining unknown quantity would have to be found; and we cannot solve an equation of higher degree than the second, except in very special cases.

For example, if we have the equations $x^2+x+y=3$ and $x^2+y^2=5$. We have from the first equation $y=3-x-x^2$; and, by substituting this value of y in the second equation, we get

$$x^2+(3-x-x^2)^2=5,$$

that is $$x^4+2x^3-4x^2-6x+4=0;$$

and this equation of the fourth degree cannot be solved by any methods which are within the range of this book.

158. We can always solve two equations of the second degree when *all the terms which contain the unknown quantities are of the second degree.* The method will be seen from the following example.

Ex. 1. Solve the equations

$$x^2+3xy=28,$$
$$xy+4y^2=8.$$

Divide the members of the first equation by the corresponding members of the second equation; we then have

$$\frac{x^2+3xy}{xy+4y^2}=\frac{28}{8}=\frac{7}{2}.$$

Hence $$2(x^2+3xy)=7(xy+4y^2);$$

$$\therefore\ 2x^2-xy-28y^2=0,$$

that is, $$(2x+7y)(x-4y)=0.$$

Hence $$x=4y, \text{ or } x=-\tfrac{7}{2}y.$$

I. If $x=4y$, we have from the second equation,

$$4y^2+4y^2=8;\ \therefore\ y=\pm 1.$$

therefore $$x=4y=\pm 4.$$

II. If $x = -\frac{7}{2}y$, we have from the second equation,

$$-\tfrac{7}{2}y^2 + 4y^2 = 8 ; \quad \therefore \; y = \pm 4.$$

And therefore $$x = -\tfrac{7}{2}y = \mp 14.$$

Thus there are four sets of values, namely $x=4, y=1$; $x=-4$, $y=-1$; $x=14, y=-4$; and $x=-14, y=4$.

We have in the above written down the factors of

$$2x^2 - xy - 28y^2$$

by inspection: when this cannot be done, the factors can be found by the method of Art. 104.

Ex. 2. Solve $x^2 - 3xy = 0$, $5x^2 + 3y^2 = 48$.

The first equation is $x(x-3y) = 0$.

Hence $$x = 0, \text{ or } x = 3y.$$

If $x=0$, the second equation gives $3y^2 = 48$; $\therefore \; y = \pm 4$.

If $x = 3y$, the second equation gives $45y^2 + 3y^2 = 48$; $\therefore \; y = \pm 1$, and then $x = \pm 3$.

159. Any pair of equations which are *homogeneous*, so far as the terms which contain the unknown quantities are concerned, can be solved by the method adopted in the last Article. Sometimes however the equations can be solved by special methods.

For example, to solve

$$x^2 + y^2 = 74,$$

$$2xy = 70.$$

By addition, we have

$$x^2 + 2xy + y^2 = 144,$$

that is $$(x+y)^2 - 12^2 = 0 ;$$

$$\therefore \; (x+y-12)(x+y+12) = 0.$$

Hence $$x + y = 12 \text{ ....(i).}$$

or $$x + y = -12 \text{ ....(ii).}$$

Again, from the given equations, we have by subtraction,

$$x^2 - 2xy + y^2 = 4,$$

that is $$(x-y)^2 - 2^2 = 0 ;$$

$$\therefore \; (x-y-2)(x-y+2) = 0.$$

Hence $x-y=2$ ....(iii),

or $x-y=-2$ ....(iv).

From (i) and (iii), we have $x=7$, $y=5$.

From (i) and (iv), we have $x=5$, $y=7$.

From (ii) and (iii), we have $x=-5$, $y=-7$.

And, from (ii) and (iv), we have $x=-7$, $y=-5$.

Thus there are four pairs of values, two of which are given by $x=\pm 7$, $y=\pm 5$, and the other two by $x=\pm 5$, $y=\pm 7$, it being understood that in both cases the upper signs are to be taken together, and the lower signs are to be taken together.

Again, taking the equations considered in Art. 158, namely

$$x^2+3xy=28,$$
$$xy+4y^2=8.$$

We have by addition $x^2+4xy+4y^2=36$,

that is $(x+2y)^2-6^2=0$;

$\therefore$ $x+2y=6$ ....(i),

or $x+2y=-6$ ....(ii).

We can now complete the solution as in Art. 156, by taking (i) and (ii) with either of the given equations.

EXAMPLES. L.

1. $x^2-2xy=0$, $4x^2+9y^2=225$.
2. $2x^2-3xy=0$, $y^2+5xy=34$.
3. $x^2+3xy=45$, $y^2-xy=4$.
4. $x^2-xy=63$, $y^2+xy=22$.
5. $x^2+xy=24$, $2y^2+3xy=32$.
6. $x^2-3xy=10$, $4y^2-xy=-1$.
7. $x^2+xy-2y^2=-44$, $xy+3y^2=80$.
8. $x^2+3xy=7$, $y^2+xy=6$.
9. $x^2-3y^2=13$, $3x^2-y^2=71$.
10. $3xy+x^2=10$, $5xy-2x^2=2$.
11. $x^2+3xy=54$, $xy+4y^2=115$.
12. $x(x+y)=40$, $y(x-y)=6$.
13. $x^2-3xy+2y^2=3$, $2x^2+y^2=6$.
14. $x^2-2xy+5=0$, $(x-y)^2-4=0$.
15. $x^2+5y^2=84$, $3x^2+17xy-y^2=-84$.
16. $x^2-7xy-9y^2=9$, $x^2+5xy+11y^2=5$.

17. $4x^2-3xy=10,$
$y^2-xy=6.$

18. $x^2+3xy=40,$
$4y^2+xy=9.$

19. $x^2+xy+y^2=7,$
$6x^2-2xy+y^2=6.$

20. $2x^2-2xy+3y^2=18,$
$3x^2-2y^2=19.$

21. $x+y+1=0,$
$\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{12}.$

22. $x+y+3=0,$
$\dfrac{1}{x}+\dfrac{1}{y}-\dfrac{1}{6}=0.$

23. $\dfrac{x}{3}+\dfrac{y}{2}=\dfrac{5}{3},$
$\dfrac{2}{x}+\dfrac{3}{y}=0.$

24. $\dfrac{x}{2}+\dfrac{y}{5}=5,$
$\dfrac{2}{x}+\dfrac{5}{y}=\dfrac{5}{6}.$

25. $(x+1)(y+1)=10,$
$xy=3.$

26. $(x+3)(y+1)=4,$
$xy+1=0.$

160. The following examples will shew how to deal with some other cases of simultaneous equations.

Ex. 1. Solve
$$x-y=2 \quad\ldots\ldots\ldots\ldots\ldots\text{(i)},$$
$$x^3-y^3=386 \quad\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

From (i) we have $x=y+2.$

Substitute this value in (ii), and we have
$$(y+2)^3-y^3=386,$$
that is $6y^2+12y+8-386=0;$
$$\therefore\ y^2+2y-63=0.$$
Hence $y=7$, or $y=-9$.

If $y=7$, $x=y+2=9.$

If $y=-9$, $x=y+2=-7.$

Thus $x=9,\ y=7$, or $x=-7,\ y=-9$.

Ex. 2. Solve
$$x-3y=2,$$
$$x^2-9y^2=8.$$

Divide the members of the second equation by the corresponding members of the first; then
$$x+3y=4;$$
and from $x+3y=4$ and $x-3y=2$,
we have $x=3,\ y=\frac{1}{3}.$

Ex. 3. Solve $$\frac{1}{x}+\frac{1}{y}=7 \text{ .... (i),}$$

$$\frac{1}{x^2}+\frac{1}{y^2}=25 \text{ .... (ii).}$$

We may consider $\frac{1}{x}$ and $\frac{1}{y}$ as the unknown quantities, as in Art. 89, Ex. 3.

Square both members of (i); then

$$\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}=49 \text{ .... (iii).}$$

From (ii) and (iii), we have

$$\frac{2}{xy}=49-25=24 \text{ .... (iv).}$$

From (ii) and (iv), we have

$$\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=25-24=1,$$

that is $$\left(\frac{1}{x}-\frac{1}{y}\right)^2=1;$$

$$\therefore \quad \frac{1}{x}-\frac{1}{y}=\pm 1 \text{ .... (v).}$$

The values of $\frac{1}{x}$ and $\frac{1}{y}$ are at once found from (i) and (v).

Ex. 4. Solve $$x^2-xy+y^2=61 \text{ .... (i),}$$

$$x^4+x^2y^2+y^4=1281 \text{ .... (ii).}$$

Divide the members of (ii) by the corresponding members of (i); we then have

$$x^2+xy+y^2=21 \text{ .... (iii).}$$

From (i) and (iii) we have, by subtraction,

$$2xy=-40,$$

or $$xy=-20 \text{ .... (iv).}$$

From (i) and (iv), we have

$$x^2-2xy+y^2=81;$$

$$\therefore \quad x-y=+9 \text{ .... (v),}$$

or $$x-y=-9 \text{ .... (vi).}$$

From (i) and (iv), we have also

$$x^2+2xy+y^2=1;$$

$$\therefore \quad x+y=1 \quad \text{.........................(vii)},$$

or $$x+y=-1 \quad \text{.....................(viii)}.$$

Then combining either of the equations (v) and (vi) with either of the equations (vii) and (viii) we get the four pairs of values $x=\pm 5$, $y=\mp 4$; $x=\pm 4$, $y=\mp 5$.

Ex. 5. Solve $$x^2-2xy=3y \quad \text{..............................(i)},$$

$$2x^2-9y^2=9y \quad \text{..............................(ii)}.$$

Multiply both sides of (i) by 3; then

$$3x^2-6xy=9y \quad \text{..............................(iii)}.$$

Hence $$2x^2-9y^2=3x^2-6xy;$$

$$\therefore \quad x^2-6xy+9y^2=0,$$

that is $$(x-3y)^2=0;$$

$$\therefore \quad x=3y.$$

Substitute in (ii); then

$$2(3y)^2-9y^2=9y,$$

or $$9y^2-9y=0;$$

$$\therefore \quad y(y-1)=0.$$

Hence $$y=0, \text{ or } y=1.$$

If $y=0$, $$x=3y=0.$$

If $y=1$, $$x=3y=3.$$

Thus $$x=0,\ y=0;$$

or else $$x=3,\ y=1.$$

Ex. 6. Solve $$x-y=2,\quad x^5-y^5=242.$$

Put $$x=z+1, \text{ then } y=z-1.$$

Hence

$$x^5-y^5=(z+1)^5-(z-1)^5$$

$$=z^5+5z^4+10z^3+10z^2+5z+1-(z^5-5z^4+10z^3-10z^2+5z-1)$$

$$=10z^4+20z^2+2.$$

Hence $$10z^4+20z^2+2=242;$$

$$\therefore\quad z^4+2z^2=24;$$

$$\therefore\quad (z^2-4)(z^2+6)=0.$$

Hence $$z=\pm 2 \text{ or } z=\pm\sqrt{-6}.$$

If $z=\pm 2$, $x=3$ or -1, and $y=1$ or -3.

If $z=\pm\sqrt{-6}$, $x=1\pm\sqrt{-6}$, and $y=-1\pm\sqrt{-6}$.

Ex. 7. Solve

$$xy+xz=27 \quad\text{(i)},$$
$$yz+yx=32 \quad\text{(ii)},$$
$$zx+zy=35 \quad\text{(iii)}.$$

From (i) and (ii), we have by addition

$$2xy+xz+yz=59 \quad\text{(iv)}.$$

From (iv) and (iii), we have by subtraction

$$2xy=24;$$

$$\therefore\quad xy=12 \quad\text{(v)}.$$

Hence, from (i), $$xz=15 \quad\text{(vi)}.$$

And, from (ii), $$yz=20 \quad\text{(vii)}.$$

From (v) and (vi), we have by multiplication

$$x^2yz=180 \quad\text{(viii)}.$$

From (vii) and (viii) we have by division

$$x^2=180\div 20=9;$$

$$\therefore\quad x=\pm 3.$$

Hence, from (v), $$y=12\div(\pm 3)=\pm 4.$$

And, from (vi), $$z=15\div(\pm 3)=\pm 5.$$

Thus $$x=\pm 3,\ y=\pm 4,\ z=\pm 5.$$

All the upper signs being taken together, and all the lower signs being taken together.

EXAMPLES. LI.

1. $x-y=3$,
$x^3-y^3=279$.

2. $x-y=2$,
$x^3-y^3=98$.

3. $x+y=7$,
$x^3+y^3=91$.

4. $x+y=1$,
$x^3+y^3=61$.

5. $x^2+xy+y^2=13$,
$x^4+x^2y^2+y^4=91$.

6. $x^2-xy+y^2=9$,
$x^4+x^2y^2+y^4=243$.

M

7. $xy(x+y)=240,\quad x^3+y^3=280.$

8. $xy(x-y)=12,\quad x^3-y^3=63.$

9. $\dfrac{1}{x}-\dfrac{1}{y}=2,\quad \dfrac{1}{x^2}+\dfrac{1}{y^2}=20.$

Y

10. $\dfrac{2}{x}+\dfrac{1}{y}=1,\quad \dfrac{2}{x^2}+\dfrac{3}{xy}+\dfrac{1}{y^2}=5.$

✓ 11. $\dfrac{3}{x^2}-\dfrac{1}{y^2}=1,\quad \dfrac{5}{x^2}-\dfrac{1}{xy}+\dfrac{2}{y^2}=3.$

12. $\dfrac{1}{x^2}-\dfrac{1}{4y^2}=3,\quad \dfrac{1}{x^2}-\dfrac{1}{xy}+\dfrac{1}{4y^2}=9.$

13. $x+y=1,\quad x^2y^2+13xy+12=0.$

✓ 14. $x+y=5,\quad 4xy=12-x^2y^2.$

15. $x^2-xy+y=-6,\quad y^2-xy-x=12.$

16. $x^2+xy-y=9,\quad y^2+xy-x=-3.$

17. $x+\dfrac{1}{y}=3,\quad y+\dfrac{1}{x}=\dfrac{4}{3}.$

18. $x+\dfrac{1}{y}=\dfrac{18}{7},\quad y+\dfrac{1}{x}=\dfrac{7}{4}.$

19. $2x^2-xy+y^2=2y,\quad 2x^2+4xy=5y.$

20. $x^3+1=9y,\quad x^2+x=6y.$

21. $2x^3+5y^3=115,\quad 3x^3+7y^3=186.$

✓ 22. $x^2y+xy^2=180,\quad x^2y^2=400.$

23. $x^4+x^2y^2+y^4=91,\quad (x^2-xy+y^2)(x-y)^2=28.$

✓ 24. $x^2+3xy+y^2+4(x+y)=13,\quad 3x^2-xy+3y^2+2(x+y)=9.$

25. $xy+\dfrac{x}{y}=\dfrac{5}{3},\quad xy+\dfrac{y}{x}=\dfrac{5}{6}.$

26. $\dfrac{y}{x}+\dfrac{1}{xy}=\dfrac{20}{3},\quad xy+\dfrac{x}{y}=\dfrac{5}{3}.$

27. $ax=by,\quad (x-a)(y-b)=ab.$

28. $a(x-a)=b(y-b),\quad xy=ax+by.$

29. $x-\dfrac{b^2}{y}=\dfrac{a^2}{x}-y=a-b.$

30. $yz=4,\qquad zx=9,\qquad xy=16.$

31. $x^2yz=6,\qquad y^2zx=12,\qquad z^2xy=18.$

32. $x(x+y+z)=8,\qquad y(x+y+z)=12,\qquad z(x+y+z)=5.$

33. $x(y+z)=6,\qquad y(z+x)=12,\qquad z(x+y)=10.$

34. $(x+y)(x+z)=2,\qquad (y+z)(y+x)=3,\qquad (z+x)(z+y)=6.$

CHAPTER XVII.

PROBLEMS.

161. We now give examples of problems in which the relations between the known and unknown quantities are expressed algebraically by means of quadratic equations.

In the solution of problems it often happens that by solving the equations which are the algebraical statements of the relations between the magnitudes of the known and unknown quantities, we obtain results which do not all satisfy the conditions of the problem.

The reason of this is that the roots of the equation are the numbers, whether *positive or negative*, *integral or fractional*, which satisfy it; but in the problem itself there may be restrictions, expressed or implied, on the numbers, and these restrictions cannot be retained in the equation. For example, in a problem which refers to a number of men, it is clear that this number must be *integral*, but this condition cannot be expressed in the equations.

Thus there are three steps in the solution of a problem. We first find the equations which are the algebraical expressions of the relations between the magnitudes of the known and unknown quantities; we then find the values of the unknown quantities which satisfy these equations; and finally we examine whether any or all of the values we have found violate any conditions which are expressed or implied in the problem, but which are not contained in the equations.

The following are examples of problems which lead to quadratic equations.

Ex. 1. How many children are there in a family, when eleven times the number is greater by five than twice the square of the number?

Let x be the number of children; then we have

$$11x = 2x^2 + 5;$$

$$\therefore \quad 2x^2 - 11x + 5 = 0,$$

that is $$(2x-1)(x-5) = 0.$$

Hence $$x = 5, \text{ or } x = \tfrac{1}{2}.$$

Thus there are 5 children, the value $\frac{1}{2}$ being inadmissible.

Ex. 2. Eleven times the number of yards in the length of a rod is greater by five than twice the square of the number of yards. How long is the rod?

This leads to the same equation as before, only in this case we cannot reject the fractional result. Thus the rod may be five yards long, or it may be half a yard long.

Ex. 3. A number of two digits is equal to twice the product of the digits, and the digit in the ten's place is less by 3 than the digit in the unit's place. What is the number?

Let x be the digit in the ten's place; then $x+3$ will be the digit in the unit's place. The number is therefore equal to

$$10x + (x+3).$$

Hence, by the question,

$$10x + (x+3) = 2x(x+3);$$

$$\therefore \quad 2x^2 - 5x - 3 = 0,$$

that is $$(x-3)(2x+1) = 0.$$

Hence $$x = 3, \text{ or } x = -\tfrac{1}{2}.$$

Now the digits of a number must be *positive integers;* hence the second value is inadmissible.

Therefore the digits are 3 and 6, and the required number is 36.

Ex. 4. The square of the number of pounds a man possesses is greater by 1000 than thirty times the number. How much is the man worth?

Let x be the number of pounds the man is worth; then, by the question, $x^2 = 30x + 1000;$

$$\therefore \quad x^2 - 30x - 1000 = 0,$$

this is $(x-50)(x+20)=0.$

Hence $x = 50$, or $x = -20$.

Both these values are admissible, provided a debt is considered as a negative possession.

Hence the man may have £50, or he may owe £20.

Ex. 5. The sum of a certain number and its square root is 42: what is the number?

Let x be the number; then

$$x + \sqrt{x} = 42,$$

that is $\sqrt{x} = 42 - x.$

Square both sides; then, after transposition, we have

$$x^2 - 85x + 1764 = 0,$$

the roots of which are 36 and 49.

The value 49 will not however satisfy the condition of the question, if by a square root of a number is meant only the *arithmetical* square root.

Ex. 6. The sum of the ages of a father and his son is 100 years; also one-tenth of the product of their ages, in years, exceeds the father's age by 180. How old are they?

Let the father be x years old; then the son will be $100 - x$ years old. Hence by the question,

$$\tfrac{1}{10}x(100 - x) = x + 180;$$

$$\therefore \quad x^2 - 90x + 1800 = 0,$$

that is $(x-60)(x-30) = 0.$

Hence $x = 60$, or $x = 30$.

The second value is inadmissible, although it is a positive integer, for it would make the son older than his father.

Hence the father must be 60, and the son 40 years old.

EXAMPLES. LII.

1. Find two numbers one of which is three times the other and whose product is 243.

2. Find two numbers whose sum is 18 and whose product is 77.

3. Find two numbers whose difference is 20, and the sum of whose squares is 650.

4. Divide 25 into two parts whose product is 156.

5. Divide 80 into two parts the sum of the squares of which is 3208.

6. A certain number is subtracted from 36, and the same number is also subtracted from 30; and the product of the remainders is 891. What is the number?

7. A rectangular court is ten yards longer than it is broad; its area is 1131 square yards. What is its length and breadth?

8. The product of the sum and difference of a number and its reciprocal is $3\frac{3}{4}$: find the number.

9. The number of fives balls which can be bought for a pound is equal to the number of shillings in the cost of 125 of them. How many can be bought for a pound?

10. The number of eggs which can be bought for one shilling is equal to the number of pence which 27 eggs cost. How many eggs can be bought for one shilling?

11. A cask contains a certain number of gallons of water, and another cask contains half as many gallons of wine; six gallons are drawn from each, and what is drawn from the one cask is put into the other. If the mixture in each cask be now of the same strength, find the amount of water and wine.

12. The cost of an entertainment was £4, which was to have been divided equally among the party, but four of them leave without paying, and the rest have each to pay one shilling extra in consequence. Of how many persons did the party consist?

13. A man buys a certain number of articles for £1, and makes £1. 1*s*. 0*d*. by selling all but two at 2*d*. each more than they cost. How many did he buy?

14. A man bought a certain number of railway-shares for £1875; he sold all but 15 of them for £1740, gaining £4 per share on their cost price; how many shares did he buy?

15. A crew can row a certain course up stream in $8\frac{4}{7}$ minutes, and if there were no stream they could row it in 7 minutes less than it takes them to drift down the stream; how long would they take to row down with the stream?

16. A boat's crew can row 8 miles an hour in still water. What is the speed of a river's current if it take them 2 hours and 40 minutes to row 8 miles up and 8 miles down?

17. Two trains run without stopping over the same 36 miles of rail. One of them travels 15 miles an hour faster than the other, and accomplishes the distance in 12 minutes less. Find the speed of the two trains.

18. A person having 7 miles to walk increases his speed one mile an hour after the first mile, and is half an hour less on the road than he would have been had he not altered his rate. How long did he take?

19. A and B together can do a piece of work in a certain time. If they each did one-half of the work separately, A would have to work one day less, and B two days more than before. Find the time in which A and B together do the work.

20. The price of photographs is raised 3*s.* per dozen; and, in consequence, seven less than before are sold for a guinea. What was the original price?

21. What are eggs a dozen when two more for a shilling would lower the price a penny a dozen?

22. A woman spends three shillings in eggs; if she had bought a dozen less for the same money, they would have cost her threepence a dozen more. How many did she buy?

23. The price of one kind of sugar per stone is 1*s.* 9*d.* more than that of another kind, and 8 lbs. less of the first kind can be got for £1 than of the second. Find the price of each per stone.

24. The number of pence which a dozen apples cost is greater by 2 than twice the number of apples which can be bought for 1*s.* How many can be bought for 9*s.*?

25. Divide £1015 among A, B and C so that B shall receive £5 less than A, and C as many times B's share as there are shillings in A's share.

26. Find two fractions whose sum is $\frac{5}{6}$, and whose difference is equal to their product.

27. Two men start at the same time to meet each other from towns which are 25 miles apart. One takes 18 minutes longer than the other to walk a mile, and they meet in 5 hours. How fast does each walk?

28. The men in a regiment can be arranged in a column twice as deep as it is broad. If the number be diminished by 206, the men can be arranged in a hollow square three deep, having the same number of men in each outer side of the square as there were in the depth of the column. How many men were there at first in the regiment?

29. The area of a certain rectangle is equal to the area of a square whose side is six inches shorter than one of the sides of the rectangle. If the breadth of the rectangle be increased by one inch and its length diminished by two inches, its area would be unaltered. Find the lengths of its sides.

30. The diagonal and the longer side of a rectangle are together five times the shorter side, and the longer side exceeds the shorter by 35 yards. What is the area of the rectangle?

31. If the greatest side of a rectangle be diminished by 3 yards and the less by 1 yard its area would be halved; and if the greater be increased by 9 yards and the less diminished by 2 yards its area would be unaltered. Find the sides.

32. Two trains A and B leave P for Q at the same time as two trains C and D leave Q for P. A passes C 120 miles from P, and D 140 miles from P. B passes C 126 miles from Q, and D half way between P and Q: find the distance from P to Q.

MISCELLANEOUS EXAMPLES: IV.

A. **1.** Simplify $2x - [3x - 9y - \{2x - 3y - (x + 5y)\}]$.

2. Multiply $a^2 + 25b^2 + 4c^2 + 5ab - 2ac + 10bc$ by $a - 5b + 2c$.

3. Divide $x^3 + (4ab - b^2)\,x - (a - 2b)\,(a^2 + 3b^2)$ by $x - a + 2b$.

4. Find the factors of (i) $(2x+y-z)^2-(x+2y+4z)^2$,
(ii) $x^2y^2-x^2-y^2+1$, and (iii) $x^2y^2z^2-x^2z-y^2z+1$.

5. Simplify (i) $\dfrac{x^3+4x^2-8x+24}{8-8x+x^3-x^4}$.

(ii) $\dfrac{x-1}{(x-2)(x-3)}-2\dfrac{x-2}{(x-3)(x-1)}+\dfrac{x-3}{(x-1)(x-2)}$.

6. Solve the equations:

(i) $\dfrac{1}{x-a}+\dfrac{2}{x-b}=\dfrac{3}{x-c}$.

(ii) $4x^2-25x-21=0$.

(iii) $x+y=\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{2}$.

7. Shew that x^2-5x+7 can never be less than $\frac{3}{4}$.

8. The difference of the cubes of two consecutive numbers is 919: find the numbers.

B. 1. Simplify $(x+3)^3-3(x+2)^3+3(x+1)^3-x^3$.

2. Shew that

$$(x-a)^2+(y-b)^2+(a^2+b^2-1)(x^2+y^2-1)=(xa+by-1)^2+(bx-ay)^2.$$

3. Divide $x^6-2a^3x^3+a^6$ by $x^2-2ax+a^2$.

4. Find the L.C.M. of $8x^3+27$, $16x^4+36x^2+81$ and $6x^2-5x-6$.

5. Simplify

(i) $\dfrac{2}{x-1}-\dfrac{1}{x+1}-\dfrac{x+3}{x^2+1}$.

(ii) $\dfrac{\{(a+b)(a+b+c)+c^2\}\{(a+b)^2-c^2\}}{\{(a+b)^3-c^3\}(a+b+c)}$.

6. Solve the equations:

(i) $(6-x)(1+2x)+3x(x+5)=(x+1)^2-x$.

(ii) $5x^2+7x=160$.

(iii) $x^2+xy=10$, $y^2-xy=3$.

7. Shew that the sum of the squares of the roots of the equation $x^2-5x+2=0$ is 21.

8. At a concert £60 was received for reserved seats, and the same sum for unreserved seats. A reserved seat cost 3 shillings more than an unreserved seat, but 360 more tickets for unreserved than for reserved seats were sold. How many tickets were sold altogether?

C. 1. Simplify $12a - 3\{b - 2(a - 3b) - 2a\}$.

2. Multiply $a^3 + 2a^2b - ab^2 + 2b^3$ by $a^3 - 2a^2b - ab^2 - 2b^3$.

3. Divide $24x^4 - 10x^3y - 8x^2y^2 + 10xy^3 - 4y^4$ by $2y^2 - xy - 4x^2$.

4. Find the factors of $9x^2 + 9x + 2$,

and of $4(ab - cd)^2 - (a^2 + b^2 - c^2 - d^2)^2$.

5. Simplify
$$\frac{x + \dfrac{y - x}{1 + xy}}{1 - \dfrac{x(y - x)}{1 + xy}},$$

and shew that $\dfrac{b - c}{a} + \dfrac{c - a}{b} + \dfrac{a - b}{c} + \dfrac{(b - c)(c - a)(a - b)}{abc} = 0$.

6. Solve the equations:

(i) $\dfrac{x - 1}{3} - \dfrac{x - 3}{5} = 7 + \dfrac{x + 2}{6}$.

(ii) $x + y = 2a,\ x^2 + y^2 = 2a^2$.

7. Find the least value of $x^2 + 6x + 12$, and the greatest value of $6x - x^2 - 4$.

8. *A* and *B* have eighteen coins between them, which are all shillings or pennies. *A* has three times as many shillings as pennies, and *B* has just as many shillings as pennies; also *A* has 9*d*. more than *B*. How much has each?

D. 1. Shew that $(b + c)^2 - a^2 + (c + a)^2 - b^2 + (a + b)^2 - c^2 = (a + b + c)^2$.

2. Arrange $(1 + x)^4 + 2(1 - x + x^2)$ according to ascending powers of x.

3. Shew that the difference between the squares of any two consecutive numbers is one more than double the smaller number.

4. Shew that if $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = 0$, $a^2 + b^2 + c^2 = (a + b + c)^2$.

5. Find the L.C.M. of

$$x^2-5x-14,\quad x^2-4x-21 \text{ and } x^3-3x^2-25x-21.$$

For what value of x will all three expressions vanish?

6. Solve the equations:

(i) $\frac{4}{x}-\frac{3}{y}+5=\frac{6}{x}+\frac{3}{y}=10.$

(ii) $\frac{1}{x}-\frac{1}{a}-\frac{1}{b}=\frac{1}{x-a-b}.$

(iii) $x+y=3,\ 3x^2-11y^2=1.$

7. If x_1, x_2 are the roots of $ax^2+bx+c=0$, prove that

$$\frac{x_1}{x_2}+\frac{x_2}{x_1}=\frac{b^2-2ac}{ac}.$$

8. Out of a cask containing 60 gallons of alcohol a certain quantity is drawn off and replaced by water. Of the mixture a second quantity, 14 gallons more than the first, is drawn off and replaced by water. The cask then contains as much water as alcohol. How much was drawn off the first time?

E. 1. Find the numerical value of

$$\frac{x}{2}-\left\{\frac{2x-3}{3}-\frac{3x-1}{4}\right\}\div\frac{x-1}{2}, \text{ when } x=4\tfrac{1}{3}.$$

2. Shew that $(a+b)^4-(a^2-b^2)^2=4ab\,(a+b)^2$, and that

$$2(a-b)(a-c)+2(b-c)(b-a)+2(c-a)(c-b)$$
$$=(b-c)^2+(c-a)^2+(a-b)^2.$$

3. Divide $(a+2b-3c+d)^2-(2a+b+3c-d)^2$ by $a+b$.

4. Find the L.C.M. of $6x^2-5ax-6a^2$ and $4x^3-2ax^2-9a^3$.

5. Simplify $\left(\frac{y^2}{x}+\frac{x^2}{y}\right)\frac{1}{y^2-x^2}+\frac{x}{xy-y^2}-\frac{y}{x^2+xy}.$

6. Solve the equations:

(i) $\frac{x+a}{x-b}+\frac{x+b}{x-a}=2.$

(ii) $(x-1)(x-2)+(x-2)(x-3)+(x-3)(x-1)=11.$

(iii) $x+\frac{1}{y}=\frac{21}{10},\qquad y+\frac{1}{x}=\frac{7}{3}.$

7. For what values of x are $\frac{x}{a}+\frac{a}{x}$ and $\frac{c}{a}+\frac{a}{c}$ equal to one another?

8. A tricyclist rode 180 miles at a uniform rate. If he had ridden 3 miles an hour slower than he did, it would have taken him 3 hours longer. How many miles an hour did he ride?

F. 1. Add together $3a^3-4a^2b+2ab^2$, $3a^2b-4ab^2+2b^3$,

$$3ab^2-4b^3+2a^3 \text{ and } 3b^3-4a^3+2a^2b.$$

2. Shew that $$(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2$$ $$=(ay-bx)^2+(bz-cy)^2+(cx-az)^2.$$

3. Divide $2+11x+11x^2+x^3-x^4$ by x^2+3x+2.

4. Find the H.C.F. of x^3-x^2-2x+2 and $x^4-3x^3+2x^2+x-1$. What value of x will make both expressions vanish?

5. Simplify (i) $$\frac{2}{x(x-2)}+\frac{1}{x^2-5x+6}+\frac{3}{x(3-x)}.$$

(ii) $$\frac{1}{1-\dfrac{1}{1-\dfrac{1}{1-x}}}.$$

6. Solve the equations:

(i) $$\frac{x+3}{5}-\frac{6-x}{10}=x-\frac{7}{10}.$$

(ii) $$\frac{x+y}{a+b}+\frac{x-y}{a-b}=2,\quad ax+by=a^2+b^2.$$

(iii) $$\frac{x+1}{x-1}+\frac{x+2}{x-2}=2\,\frac{x+3}{x-3}.$$

7. Find the difference of the squares of the roots of the equation

$$x^2-7x+9=0.$$

8. What number exceeds its square root by 156?

MISCELLANEOUS EQUATIONS.

1. $(x-1)(x-4)+(x-3)(x-5)=2(x-3)^2$.
2. $\frac{1}{2}(x-2)-\frac{1}{3}(x-4)+\frac{1}{4}(x-5)=0$.
3. $\dfrac{1}{x-3}-\dfrac{1}{x-5}=\dfrac{1}{x-2}-\dfrac{1}{x-4}$.
4. $15x+17y=79,\quad 17x+15y=81$.

5. A man drives to a certain place at the rate of 8 miles an hour. Returning by a road 3 miles longer at the rate of 9 miles an hour, he takes $7\frac{1}{2}$ minutes longer than in going. How long is each road?

---

6. $\frac{1}{2}(x-2)-\frac{x-5}{9}+5\frac{x-1}{6}=0.$

7. $3(x+1)(x-3)+(x+7)(3x-13)=(2x-5)(3x-3).$

8. $\frac{2x+9}{x+2}+\frac{3x-2}{x+3}=\frac{5x+14}{x+4}.$

9. $\frac{x}{7}+\frac{y}{8}=82,\ \frac{x}{8}+\frac{y}{7}=83.$

10. A pound of tea and 5 pounds of sugar cost 3*s.* 9*d.*, but if the tea were to rise 20 per cent. and the sugar 25 per cent. in price, they would cost 4*s.* 7*d.* What was the price of the tea?

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11. $\frac{1}{9}(6x+3)+\frac{1}{11}(7x+6)+\frac{1}{13}(9x+2)=2x+1.$

12. $\frac{x}{a}+\frac{x}{b}=\frac{a}{b}-\frac{b}{a}.$

13. $\frac{x+3}{x+1}+\frac{x-6}{x-4}=\frac{x+4}{x+2}+\frac{x-5}{x-3}.$

14. $x+y=2,\ (a+b)x+(b-a)y=2b.$

15. A poultry keeper obtained 100 eggs more in May than in April, and the daily average in May was 3 more than in April. How many did he have in May?

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16. $3x-3[4x-2(2x-5)]=9-2[3x-5(x-5)].$

17. $\frac{5}{x}-\frac{6}{y}=2,\ \frac{4}{x}+\frac{12}{y}=3.$

18. $15x^2+34x+15=0.$

19. $\frac{x+2}{x-4}-\frac{x-3}{x-5}=1.$

20. A father's age is three times that of his youngest son, and in 7 years he will be twice as old as his oldest son; also the oldest son is 5 years older than the youngest. What are their present ages?

---

21. $\frac{1}{2}(x+1)(x+3)=\frac{1}{3}(x+5)(x+2)+\frac{1}{6}(x-1)(x-4)$.

22. $(b+c)(x-a)-(c+a)(x-b)=(a+b)(x-c)$.

23. $17x-19y=4,\ 27x-29y=24$.

24. $\frac{2x-3}{x-3}+\frac{3x+8}{2x+1}=5$.

25. Two passengers, who have altogether 360 lbs. of luggage, are charged 3*s*. 4*d*. for excess above the weight allowed free; if one passenger had taken the same weight of luggage he would have been charged 5*s*. 5*d*. What weight is allowed free?

---

26. $\frac{3x-1}{5}+\frac{5x+1}{6}=\frac{9x+1}{8}-\frac{1-x}{3}$.

27. $\frac{x+y}{3}=2x-y-2,\ \frac{2y-x}{2}=y-x+5$.

28. $(a+b)x+(a-b)y=a^2-b^2,$
$(a-b)x+(a+b)y=a^2+b^2$.

29. $\frac{3}{3-x}+\frac{8}{8-x}=3$.

30. A father's age is equal to the united ages of his five children, and five years ago his age was double their united ages. How old is the father?

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31. $\frac{2x}{x-1}+\frac{x+3}{x+1}=\frac{3x+19}{x+5}$.

32. $\frac{1}{5}(x-2)=\frac{1}{4}(1-y),\ 26x+3y+4=0$.

33. $\frac{1}{x^2-1}-\frac{1}{1-x}=\frac{7}{8}-\frac{1}{x+1}$.

34. $\sqrt{x}+\sqrt{5x+1}=2$,

35. Two cyclists started at the same time, and one rode from *B* to *A* and the other from *A* to *B* along the same road. They reached their destinations 3 hours and 1 hour 20 minutes respectively after they passed each other. When did they meet?

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36. $\frac{5x+8}{5x-1}+\frac{9}{1-25x^2}=\frac{1+10x}{1+5x}$.

37. $3y-\frac{2}{x}=5y-\frac{3}{x}+9=4y-\frac{7}{x}$.

38. $\frac{4x-7}{3x-4}+\frac{2x+1}{x-7}=18\tfrac{1}{4}$.

39. $\sqrt{4x+4}-\sqrt{x+8}=\sqrt{x-4}$.

40. A certain sum of money was divided between *A*, *B* and *C*. *A* had one-ninth of the shares of *B* and *C* together, *B* had one-third of the shares of *C* and *A* together, and *C* had 6 shillings more than *A* and *B* together. How much had each?

41. $\frac{x}{4}\left(3-\frac{8}{x}\right)-\left(7-\frac{3x}{4}\right)=5-\frac{15}{64}x$.

42. $\frac{x+3}{x-3}-\frac{x-2}{x-4}=1$.

43. $x+2=\sqrt{\{4+x\sqrt{8-x}\}}$.

44. $3x-2y=12,\ 9x^2-4y^2=576$.

45. A man about to invest in the $2\frac{3}{4}$ per cents. observed that, if the price of the stock had been $7\frac{1}{3}$ less, he would have got $\frac{2}{9}$ per cent. more interest for his money. What was the price of the stock?

46. $\frac{4x^3+4x^2+8x+1}{2x^2+2x+3}=\frac{2x^2+2x+1}{x+1}$.

47. $\frac{1}{10}(5x+6)-\frac{1}{21}(11y-5)=11,\ \frac{1}{25}(55y-12)=\frac{7x}{5}-37$.

48. $\frac{12}{x}+\frac{10}{x-1}=6\tfrac{1}{3}$.

49. $\sqrt{x}+\sqrt{x+4}=\frac{2}{\sqrt{x}}$.

50. A fast train left Cambridge for London at the same time as a slow train left London for Cambridge. They arrived at their destinations three-quarters of an hour and one hour and twenty minutes respectively after they passed one another. Find how long each took for the journey.

51. $\dfrac{x-\frac{1}{2}}{x-1} - \dfrac{3}{5}\left(\dfrac{1}{x-1} - \dfrac{1}{3}\right) = \dfrac{23}{10(x-1)}$.

52. $5x + 2y - 1 = 3x - y + 14 = x + 19y + 6$.

53. $\sqrt{(x^2 - a^2)} = 2 - \dfrac{1}{\sqrt{(x^2 - a^2)}}$.

54. $3x^2 - 2xy + 4y^2 = 36,\ 4x^2 - y^2 = 7$.

55. Find three consecutive integers such that the sum of the cubes of the greatest and least exceeds twice the cube of the middle number by 42.

56. $\dfrac{x-\frac{1}{2}}{3} - \dfrac{\frac{x}{3} - 12}{5} = \dfrac{2x-1}{3} + \dfrac{9\frac{1}{2} - x}{21}$.

57. $\dfrac{x-3}{2} + \dfrac{x-2}{3} = \dfrac{2}{x-3} + \dfrac{3}{x-2}$.

58. $ax - by = a^2 - b^2 - 2ab,\ bx + ay = 2ab + a^2 - b^2$.

59. $x^2 + xy + y^2 = 9,\ x^4 + x^2y^2 + y^4 = 243$.

60. A man bought a certain number of articles, of which he sold half at 5 per cent. profit, one-third at 10 per cent. profit, and the rest at 2*s*. 3*d*. each. His total profit was at the rate of $1\frac{2}{3}$ per cent. What was the cost of each article?

61. $(a+b)(x+a-b) + (a-b)(x-a-b) + 2a(x-2c) = 0$.

62. $\dfrac{3x-2}{2x-5} - \dfrac{2x-3}{3x-2} = \dfrac{2}{3}$.

63. $2x^2 - 3\sqrt{(x^2 + 2x + 14)} + 4x - 49 = 0$.

64. $2x^2 + 3xy = 8,\ y^2 - 2xy = 20$.

65. A man bought a certain number of sheep for £58. Having lost five of them, he sold one quarter of the remainder for 12 guineas making a profit of 5 per cent. on the sale of these. How many did he buy?

66. $\dfrac{1}{5x-1} - \dfrac{x+5}{1-25x^2} = \dfrac{3}{1+5x}$.

67. $(a+b)x + by = ax + (a+b)y = a^3 + b^3$.

68. $\sqrt{(x-a)}+\sqrt{(x-b)}=\dfrac{b}{\sqrt{(x-a)}}+\dfrac{a}{\sqrt{(x-b)}}$.

69. $x^2+y^2=xy+7,\ x^2-y^2=xy-1$.

70. A number of articles were bought for £5, and sold again at an advance of 6*d.* in the price of each; and at the advanced price £5 was received for the sale of all but 10 of the articles. How many were there?

71. $\frac{1}{5}(3x+1)-\frac{1}{3}(2x-3)=\frac{1}{10}(6x-1)-\frac{1}{6}(7x-3)$.

72. $\dfrac{x}{c}-\dfrac{c}{x}=\dfrac{c}{a}-\dfrac{a}{c}$.

73. $\sqrt{(x+1)}+\sqrt{2x}=\sqrt{(6x+1)}$.

74. $x^2-xy+3y=11,\ y^2-xy-3x+1=0$.

75. The value of 150 coins consisting of half-crowns, shillings and sixpences amounts to £10. 16*s*. If there were twice as many sixpences, half as many half-crowns, and three times as many shillings, the total value of the coins would be £15. 11*s*. How many coins are there of each kind?

76. $\dfrac{3x+1}{5}-\dfrac{1-5x}{3}=\dfrac{x+1}{3}-\dfrac{2-9x}{7}$.

77. $\dfrac{x+1}{x^2+1}+\dfrac{x^2+1}{x+1}=\dfrac{29}{10}$.

78. $(x-a)(y-b)=ab,\ bx=ay$.

79. $x^2=2y+24,\ y^2=2x+24$.

80. 1120 square feet of paper will just cover the four walls of a room which is 8 feet longer than it is wide; but, if the room were 4 feet higher, the same quantity of paper would just cover the two smaller and one of the larger walls. What are the dimensions of the room?

81. $\dfrac{1}{2}(x-2)-\dfrac{x-5}{9}+5\dfrac{x-1}{6}=0$.

82. $(a-x)^3+(b-x)^3=(a+b-2x)^3$.

83. $(x-5)^2+(y-6)^2=2(xy-40),\ x=y+1$.

84. $4xy=96-x^2y^2,\ x+y=6.$

85. A merchant gained as many sovereigns on the sale of a certain quantity of coal as there were shillings in the cost price of a ton, or pence in the retail price of a cwt. How many tons did he sell?

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86. $\dfrac{2x-3}{2x-4}-\dfrac{2x-4}{2x-5}=\dfrac{2x-6}{2x-7}-\dfrac{2x-7}{2x-8}.$

87. $2x(a-b-2x)+(a-1)(b-1)=0.$

88. $\sqrt{(6x+1)}+\sqrt{2(1-x)}=\sqrt{7x+6}.$

89. $xy(x+y)=30,\ x^3+y^3=-91.$

90. A rectangular enclosure is half an acre in area, and its perimeter is 198 yards. Find the length of its sides.

---

91. $\dfrac{x+3}{2x+1}+\dfrac{3x+7}{2x-2}=\dfrac{2x+1}{x-2}.$

92. $x+2y+3z=4,\ x+3y+2=4z,\ x+2z+3=4y.$

93. $\sqrt[3]{(a-x)}+\sqrt[3]{(x-b)}=\sqrt[3]{(a-b)}.$

94. $(x-3)^2+(y-3)^2=34,\ xy-3(x+y)=6.$

95. Two trains start at the same instant, the one from B to A, the other from A to B, the distance between A and B being 100 miles. The trains meet in 1 hour 15 minutes, and one train gets to its destination 1 hour 20 minutes before the other. Find the rates of the trains.

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96. $\dfrac{ax-by}{3}=\dfrac{(2a+b)x-(a+2b)y}{7}=a^2-b^2.$

97. $13x^2+12=80x.$

98. $(x+1)(x+2)(x+4)(x+5)=4.$

99. $x^2(y+1)+y^2(x+1)=109,\ xy=12.$

100. A starts to walk from P to Q at 10 A.M., B starts from Q to walk to P at 10.24 A.M. They meet 6 miles from Q. B stops 1 hour at P and A stops 2 hrs. 54 min. at Q, and returning they meet midway between P and Q at 6.54 P.M. Find the distance from P to Q.

CHAPTER XVIII.

POWERS AND ROOTS. SQUARE ROOT.

162. THE process by which the powers of quantities are obtained is often called **involution**; and the reverse process, namely that by which the roots of quantities are obtained, is called **evolution**.

We shall in the present chapter consider some cases of involution, and of evolution.

163. When m and n are any positive integers, we have by definition

$$a^m = aaaa\ldots \text{ to } m \text{ factors,}$$

and $$a^n = aaaa\ldots \text{ to } n \text{ factors;}$$

$$\therefore\ a^m \times a^n = (aaaa\ldots \text{ to } m \text{ factors}) \times (aaaa\ldots \text{ to } n \text{ factors})$$
$$= aaaa\ldots \text{ to } m+n \text{ factors}$$
$$= a^{m+n}, \qquad \text{by definition.}$$

Hence when m and n are any positive integers,

$$a^m \times a^n = a^{m+n}.$$

Thus *the index of the product of any two powers of the same quantity is the sum of the indices of the factors.* This result is called the **Index Law.**

From the Index Law we have

$$a^m \times a^n \times a^p = a^{m+n} \times a^p = a^{m+n+p},$$

and so on, however many factors there may be.

Hence $$a^m \times a^n \times a^p \ldots\ldots = a^{m+n+p+\cdots}.$$

Thus, *the index of the product of any number of powers of the same quantity is the sum of the indices of the factors.*

164. We have

$$a^m \div a^n = \frac{a \times a \times a \times a \ldots \text{ to } m \text{ factors}}{a \times a \times a \times a \ldots \text{ to } n \text{ factors}}.$$

Now, if m is greater than n, the n factors of the denominator can be cancelled with n of the factors of the numerator: we then have $m-n$ factors left in the numerator.

Thus, when m is greater than n,

$$a^m \div a^n = a^{m-n}.$$

If however n is greater than m, the m factors of the numerator can be cancelled with m of the factors of the denominator: we then have $n-m$ factors left in the denominator.

Thus, when m is less than n,

$$a^m \div a^n = \frac{1}{a^{n-m}}.$$

165. To find $(a^m)^n$ when m and n are positive integers.

By definition

$$\begin{aligned}(a^m)^n &= a^m \times a^m \times a^m \times \ldots \text{ to } n \text{ factors}\\ &= a^{m+m+m+\ldots \text{ to } n \text{ terms}} \qquad \text{, by Art. 163.}\\ &= a^{mn}.\end{aligned}$$

Hence $$(a^m)^n = a^{mn}.$$

Thus, *to raise any power of a quantity to any other power, its original index must be multiplied by the index of the power to which it is to be raised.*

166. To find $(ab)^m$.

$$\begin{aligned}(ab)^m &= ab \times ab \times ab \ldots\ldots \text{ to } m \text{ factors}, \qquad \text{by definition,}\\ &= (aaa\ldots \text{ to } m \text{ factors}) \times (bbb\ldots \text{ to } m \text{ factors})\end{aligned}$$

[Art. 39.

$$= a^m \times b^m, \qquad \text{by definition.}$$

Hence $$(ab)^m = a^m b^m.$$

Similarly

$$\begin{aligned}(abc)^m &= abc \times abc \times abc \ldots \text{ to } m \text{ factors,}\\ &= (aaa\ldots \text{ to } m \text{ factors}) \times (bbb\ldots \text{ to } m \text{ factors})\\ &\qquad\qquad\qquad\qquad\qquad \times (ccc\ldots \text{ to } m \text{ factors})\\ &= a^m \times b^m \times c^m.\end{aligned}$$

Hence $(abc)^m = a^m b^m c^m$,

and so on, however many factors there may be in the expression whose mth power is required.

Thus, *the* m*th power of* a *product is the product of the* m*th powers of its factors.*

167. The most general monomial expression is of the form $a^x b^y c^z \ldots$.

To find $(a^x b^y c^z \ldots)^m$.

$$\begin{aligned}(a^x b^y c^z \ldots)^m &= (a^x)^m (b^y)^m (c^z)^m \ldots && \text{[Art. 166.]}\\ &= a^{xm} b^{ym} c^{zm} \ldots && \text{[Art. 165.]}\end{aligned}$$

Thus *any power of a monomial expression is obtained by taking each of its factors to a power whose index is the product of its original index and the index of the power to which the whole expression is to be raised.*

168. The following is an important case.

To find $\left(\frac{a}{b}\right)^m$.

$$\begin{aligned}\left(\frac{a}{b}\right)^m &= \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \ldots\ldots \text{ to } m \text{ factors,} && \text{by definition,}\\ &= \frac{a \times a \times a \ldots \text{ to } m \text{ factors}}{b \times b \times b \ldots \text{ to } m \text{ factors}} && \text{[Art. 134.]}\\ &= \frac{a^m}{b^m}.\end{aligned}$$

$$\therefore \quad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

169. It should be noticed that all powers of a positive quantity are positive, and that successive powers of a negative quantity are alternately positive and negative. This follows at once from the Law of Signs; for we have

$$(-a)^2 = (-a)\ (-a) = +a^2;$$

$$(-a)^3 = (-a)^2 (-a) = (+a^2)(-a) = -a^3;$$

$$(-a)^4 = (-a)^3 (-a) = (-a^3)(-a) = +a^4;$$

and so on.

Thus $(-a)^{2n} = +a^{2n}$, and $(-a)^{2n+1} = -a^{2n+1}$.

From the above it is clear that all even powers, whether of positive or of negative quantities, are positive; and that all odd powers of any quantity have the same sign as the original quantity.

170. We have already proved the following cases of the involution of binomial expressions.

$$(a+b)^2 = a^2 + 2ab + b^2,$$

and $$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

If we multiply again by $a+b$, we shall have

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

By multiplying the last result by $a+b$ we should obtain $(a+b)^5$, and by continuing the process we could obtain any required power of $a+b$; but to find in this way any high power, for instance to find $(a+b)^{20}$, would clearly be very laborious.

We shall in a subsequent chapter prove a theorem, called the Binomial Theorem, which will enable us to write down at once any power of a binomial expression.

Since the above formulæ are true for *all* values of a and b, we can write down the squares and the cubes of any binomial expressions.

Thus
$$\begin{aligned}(a^4-b^4)^2&=(a^4)^2+2(a^4)(-b^4)+(-b^4)^2\\&=a^8-2a^4b^4+b^8,\end{aligned}$$
and
$$\begin{aligned}(5a^2-3b^2)^3&=(5a^2)^3+3(5a^2)^2(-3b^2)\\&\quad+3(5a^2)(-3b^2)^2+(-3b^2)^3\\&=125a^6-225a^4b^2+135a^2b^4-27b^6.\end{aligned}$$
Also
$$\begin{aligned}(a+b+c)^3&=\{a+(b+c)\}^3\\&=a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3.\end{aligned}$$

171. An important case of involution is considered in Art. 56, where the square of a multinomial expression is obtained.

EXAMPLES. LIII.

Write down the value of each of the following:

1. $(a^3)^5$.
2. $(a^5)^3$.
3. $(-a^2)^3$.
4. $(-a^3)^2$.
5. $(-2a^5)^4$.
6. $(-3a^4)^5$.
7. $(-ab^2)^4$.
8. $(a^3b^4)^5$.
9. $(-ab^4)^7$.
10. $(-3a^7b^5c)^3$.
11. $(-ab^2c^5)^4$.
12. $(-5a^2b^3c^4)^3$.
13. $\left(\frac{a^2}{b^2}\right)^3$.
14. $\left(-\frac{a^3}{bc^2}\right)^5$.
15. $\left(-\frac{a}{b^2c^3}\right)^6$.
16. $(2a^4+3b^3)^2$.
17. $(a^5-2b^4)^2$.
18. $(-ax^4+by^3)^2$.
19. $(-a^2+2ab)^2$.
20. $(a^2+b^2)^3$.
21. $(a^3+b^3)^3$.
22. $(2a^2-3b^2)^3$.
23. $(3a^2-2b^2)^3$.
24. $(a^2+b^2+c^2)^2$.
25. $(a^3-2b^3+3c^3)^2$.
26. $(a^2-4b^2-3c^2)^2$.
27. $(x^2-3x-6)^2$.
28. $(3x^2-x-5)^2$.
29. $(2x^2+5x-1)^2$.
30. $(3x^2-6x-6)^2$.
31. $(1+x+x^2+x^3)^2$.
32. $(x^3-x^2+x-1)^2$.
33. $(x^3+x^2-2x-2)^2$.
34. $(a+2b+3c+4d)^2$.
35. $(2a-b+c-2d)^2$.
36. $(x^2+x+1)^3$.
37. $(x^2-x+2)^3$.
38. $(3x^2-5x+1)^3$.

172. We shall now consider some cases of **evolution**.

We know that there are *two* square roots of a^2, namely $\pm a$; we also know [Art. 155] that there are *three* cube roots of a^3, of which a is one, and the other two are imaginary.

There is therefore an important difference between powers and roots; for there is only one nth *power*, but there is more than one nth *root*, of a given expression.

173. An expression which when raised to the nth power, where n is any positive integer, becomes equal to a given expression, is called *an* nth *root* of the given expression.

We have shewn in Art. 166 that the mth power of a product is the product of the mth powers of its factors; hence, conversely, the mth root of a product is the product of the mth roots of its factors*.

Thus $$\sqrt{abc}=\sqrt{a}\sqrt{b}\sqrt{c},$$

and $$\sqrt[m]{ab}=\sqrt[m]{a}\sqrt[m]{b}.$$

Again, we have shewn in Art. 167 that the nth *power* of a monomial expression is obtained by *multiplying* the index of each of its factors by n.

It follows conversely that if we *divide* the index of each factor of a given expression by n we shall obtain an nth *root* of the expression. For by raising to the nth power the result obtained by such division of the indices, we must clearly get the original expression.

---

* It should be noticed that the proof, in Art. 39, that the factors of a product may be taken in any order, only holds good when those factors represent integral or fractional numbers, and does not enable us to assert that $\sqrt{a}\times\sqrt{b}=\sqrt{b}\times\sqrt{a}$, when $\sqrt{a}$ or $\sqrt{b}$ is really a surd. For a proof that surds obey the Fundamental Laws of Algebra see *Treatise on Algebra*, Art. 162.

Thus one value of $\sqrt[2]{a^4}$ is a^2, one value of $\sqrt[3]{a^6b^{12}c^9}$ is $a^2b^4c^3$, and one value of $\sqrt[n]{a^{np}b^{nq}}$ is a^pb^q.

When the square root of an expression which is not a perfect square, or the cube root of an expression which is not a perfect cube, is required, the operation cannot be performed. We can, for example, only write the square root of a as $\sqrt{a}$, and the cube root of a^2 as $\sqrt[3]{a^2}$, and similarly in other cases.

SQUARE ROOT.

174. We now proceed to consider the square root of multinomial expressions.

In Art. 96 we have shewn how to write down the square root of any *trinomial* expression which is a complete square.

Having arranged the expression according to ascending or descending powers of some letter, the square root of the whole expression is then found by taking the square roots of the extreme terms with the same or with different signs according as the sign of the middle term is positive or negative.

Thus, to find the square root of

$$4a^8-12a^4b^3+9b^6.$$

The square roots of the extreme terms are

$$\pm 2a^4 \text{ and } \pm 3b^3.$$

Hence, the middle term being negative, the required square root is $$\pm(2a^4-3b^3).$$

Note. In future only one of the square roots of an expression will be given, namely that one for which the sign of the first term is positive: to find the other root all the signs must be changed.

As other examples

$$\sqrt{(49a^{10}+28a^5+4)}=7a^5+2,$$

$$\sqrt{(1+5xy^3+\tfrac{25}{4}x^2y^6)}=1+\tfrac{5}{2}xy^3,$$

and $$\sqrt{\{a^2+2a\,(b+c)+(b+c)^2\}}=a+b+c.$$

175. When an expression which contains only two different powers of a particular letter is arranged according to ascending or descending powers of that letter, it will only contain *three* terms. For example, the expression

$$a^2+b^2+c^2+2bc+2ca+2ab,$$

which contains no other power of a but a^2 and a, when arranged according to powers of a, is

$$a^2+2a\,(b+c)+(b^2+c^2+2bc).$$

Thus any expression which only contains two different powers of a particular letter can be written as a *trinomial* expression; and since we can write down the square root of *any* trinomial expression which is a complete square, it follows that the square root of any expression which is a complete square can be written down by *inspection*, provided that the expression only contains *two different powers of some particular letter.*

For example, to find the square root of

$$a^2+b^2+c^2+2bc+2ac+2ab.$$

Arranging the expression according to descending powers of a, we have $$a^2+2a\,(b+c)+(b^2+2bc+c^2),$$

that is, $a^2+2a(b+c)+(b+c)^2$,

which is $\{a+(b+c)\}^2$.

Thus $\sqrt{}(a^2+b^2+c^2+2bc+2ca+2ab)=(a+b+c)$.

Also, to find the square root of

$$x^4+4y^4+9z^4+4x^2y^2-6x^2z^2-12y^2z^2.$$

Arrange the expression according to descending powers of x; we then have

$$x^4+2x^2(2y^2-3z^2)+4y^4+9z^4-12y^2z^2,$$

that is, $x^4+2x^2(2y^2-3z^2)+(2y^2-3z^2)^2$,

which is $\{x^2+(2y^2-3z^2)\}^2$.

Thus $\sqrt{}(x^4+4y^4+9z^4+4x^2y^2-6x^2z^2-12y^2z^2)=(x^2+2y^2-3z^2)$.

Again, to find the square root of

$$a^2+2abx+(b^2+2ac)x^2+2bcx^3+c^2x^4.$$

Arrange the expression according to powers of a; we then have

$$a^2+2a(bx+cx^2)+b^2x^2+2bcx^3+c^2x^4,$$

that is, $a^2+2a(bx+cx^2)+(bx+cx^2)^2$,

which is $\{a+(bx+cx^2)\}^2$.

Hence the required square root is $(a+bx+cx^2)$.

And to find the square root of

$$x^6-2x^5+3x^4+2x^3(y-1)+x^2(1-2y)+2xy+y^2.$$

The expression only contains y^2 and y; we therefore arrange it according to powers of y; we then have

$$y^2+2y(x^3-x^2+x)+x^6-2x^5+3x^4-2x^3+x^2.$$

Now if the expression is a complete square at all, the last of the three terms must be the square of half the coefficient of y; and it is easy to verify that

$$x^6-2x^5+3x^4-2x^3+x^2 \text{ is } (x^3-x^2+x)^2.$$

Thus the given expression is

$$y^2+2y(x^3-x^2+x)+(x^3-x^2+x)^2.$$

The required square root is therefore

$$y+x^3-x^2+x.$$

From the above it will be seen that however many terms there may be in an expression which is a perfect square, the square root can be written down *by inspection*, provided only that the expression contains only *two different powers of some particular letter*.

EXAMPLES. LIV.

Write down the square roots of the following expressions.

1. $9x^2-30xy+25y^2$.

2. $25x^4-30x^2y^2+9y^4$.

3. $4x^4-12x^2y^2+9y^4$.

4. $4x^{10}-12x^5y^3+9y^6$.

5. $x^8-6x^4y^4+9y^8$.

6. $9x^{12}-6x^6y^3+y^6$.

7. $\frac{1}{4}x^6-\frac{1}{3}x^3y^3+\frac{1}{9}y^6$.

8. $\frac{1}{9}x^8y^6-\frac{1}{3}x^4y^3+\frac{1}{4}$.

9. $25x^8y^6-40a^2b^3x^4y^3+16a^4b^6$.

10. $\frac{x^4}{a^2}+8x^2y^3+16a^2y^6$.

11. $9\frac{b^2x^4}{a^2}-24x^2y^2+16\frac{a^2y^4}{b^2}$.

12. $\frac{9}{x^6}-42\frac{a^5}{x^3}+49a^{10}$.

13. $a^2+4b^2+9c^2+12bc+6ca+4ab$.

14. $4a^2+b^2+9c^2+6bc-12ca-4ab$.

15. $4a^4+b^4+c^4-2b^2c^2-4c^2a^2+4a^2b^2$.

16. $25a^4+9b^4+4c^4-12b^2c^2+20c^2a^2-30a^2b^2$.

176. In order to shew how to find the square root of *any* algebraical expression, we will take an expression and form its square, and then shew how to reverse the process.

Consider, for example,

$$x^2+2xy+3y^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

whose square is

$$x^4+4x^3y+10x^2y^2+12xy^3+9y^4 \ldots\ldots\ldots\ldots\text{(ii)},$$

both expressions being arranged according to descending powers of x.

We may write the square of $x^2+2xy+3y^2$ in either of the following forms :

$$\{x^2+(2xy+3y^2)\}^2=x^4+2x^2(2xy+3y^2)+(2xy+3y^2)^2\ldots\text{(iii)},$$
$$\{(x^2+2xy)+3y^2\}^2=(x^2+2xy)^2+2(x^2+2xy)3y^2+(3y^2)^2\ldots\text{(iv)}.$$

Now it is clear that the first term of (ii) is the square of the first term of (i). Hence the *first term* of the root of (ii) is found by taking the square root of its first term.

Again, we see from (iii) that when we have subtracted x^4 (the square of the first term of the root) the term in the remainder which contains the highest power of x is $2x^2 \times 2xy$, which is twice the product of the first and second terms of the root.

Hence, after subtracting from (ii) the square of the first term of the root, the *second term* is obtained by *dividing the first term of the remainder by twice the first term of the root.*

Again, we see from (iv) that when we have subtracted $(x^2+2xy)^2$, that is the square of the part of the root already found, the term in the remainder which contains the highest power of x is $2x^2 \times 3y^2$, which is twice the product of the first and third terms of the root.

Hence, after subtracting from (ii) the square of that part of the root already found, the *next term* of the root is obtained by *dividing the first term of the remainder by twice the first term of the root.*

If we now subtract the square of $x^2+2xy+3y^2$ from the given expression there will be no remainder; and hence $x^2+2xy+3y^2$ is the required root.

We will now consider the most general case.

Suppose we have to find the square root of $(A+B)^2$, where A stands for any number of terms of the root, and B for the rest; the terms in A and B being *arranged according to descending* (*or ascending*) *powers of some letter,* *so that every term in* A *is of higher* (*or lower*) *dimensions than any term in* B.

Also suppose that the terms in A are known, and that we have to find the terms in B.

Subtracting A^2 from $(A+B)^2$, we have the remainder $(2A+B)\,B$.

Now from the mode of arrangement it follows that the term of the highest (or lowest) degree in the remainder is twice the product of the *first* term in A and the *first* term in B.

Hence, to obtain the *next term* of the required root, that is, to obtain the highest (or lowest) term of B, we *subtract from the whole expression the square of that part of the root which is already found, and divide the first term of the remainder by twice the first term of the root.*

The first term of the root is clearly the square root of the first term of the given expression; and when we have found the first term of the root we can find each of the other terms in succession by the above process.

For example, to find the square root of

$$x^4+4x^3y+10x^2y^2+12xy^3+9y^4.$$

$$\begin{array}{rl}
 & \underline{x^4+4x^3y+10x^2y^2+12xy^3+9y^4}\;(x^2+2xy+3y^2 \\
(x^2)^2 = & \underline{x^4\qquad\qquad\qquad} \\
(x^2+2xy)^2 = & \underline{x^4+4x^3y+4x^2y^2} \\
(x^2+2xy+3y^2)^2 = & \underline{x^4+4x^3y+10x^2y^2+12xy^3+9y^4}.
\end{array}$$

The given expression must first be *arranged according to ascending or descending powers of some letter.*

We then take the square root of the first term of the given expression: we thus obtain x^2, the *first term* of the required root.

Now subtract the square of x^2 from the given expression, and divide the first term of the remainder by $2x^2$: we thus obtain $2xy$, the *second term* of the root.

Now subtract the square of x^2+2xy, which is

$$x^4+4x^3y+4x^2y^2,$$

from the given expression, and divide the first term of the remainder, namely $6x^2y^2$, by $2x^2$: we thus obtain $3y^2$, the *third term* of the root.

Subtract the square of $x^2+2xy+3y^2$ from the given expression and there is no remainder.

Hence $x^2+2xy+3y^2$ is the required square root.

The squares of x^2, x^2+2xy, &c. are placed under the given expression, like terms being placed in the same column: the remainder left after taking away any square is then obvious by inspection.

177. Instead of finding each square independently, some labour can be saved by making use of the previous square. Thus the process of finding the square root of an algebraical expression is generally written as follows, the same example being taken as before

$$\begin{array}{r|lllll}
 & x^4 + & 4x^3y + & 10x^2y^2 + & 12xy^3 + & 9y^4 \;(\,x^2+2xy+3y^2 \\
 & x^4 & & & & \\ \cline{2-2}
2x^2+2xy\;) & & 4x^3y & & & \\
 & & 4x^3y + & 4x^2y^2 & & \\ \cline{3-4}
2x^2+4xy+3y^2\;) & & & 6x^2y^2 & & \\
 & & & 6x^2y^2 + & 12xy^3 + & 9y^4 \\ \cline{4-6}
\end{array}$$

The first term of the root is x^2. Having subtracted the square of x^2, namely x^4, the first term of the remainder is $4x^3y$.

Now *double the part of the root already found* and divide the first term of the remainder by the result; we thus obtain *the next term of the root*, namely $2xy$. *Add this new term of the root to* $2x^2$, placing the sum in the ordinary position for the divisor. Now *multiply the sum by* $2xy$ and subtract the product from the remainder $4x^3y+\ldots$, we then have the remainder $6x^2y^2+\ldots$. Repeat this process as often as may be necessary.

[It is of importance to notice that by the above process we have at the end of the second stage subtracted altogether $(x^2+2xy)^2$, since $(x^2+2xy)^2=(x^2)^2+(2x^2+2xy)\times 2xy$. Similarly at the end of every stage it will be seen that what is subtracted on the whole is the square of that part of the root which has been found up to that stage.]

EXAMPLES. LV.

Find the square roots of the following expressions:—

1. $x^4+2x^3+3x^2+2x+1.$

2. $4x^4-8x^3+4x+1.$

3. $9x^4-36x^3+72x+36.$

4. $1-xy-\frac{15}{4}x^2y^2+2x^3y^3+4x^4y^4.$

5. $4x^4+4x^3-\frac{1}{2}x+\frac{1}{16}.$

6. $x^4-2x^3+\frac{3}{2}x^2-\frac{1}{2}x+\frac{1}{16}.$

7. $x^4+2x^3y+3x^2y^2+2xy^3+y^4.$

8. $16-96x+216x^2-216x^3+81x^4.$

9. $1+4x+10x^2+12x^3+9x^4.$

10. $4x^4-4x^3+3x^2-x+\frac{1}{4}.$

11. $(1+2x^2)^2-4x(1-x)(1+2x).$

12. $x^6-4x^5+6x^4-8x^3+9x^2-4x+4.$

13. $9x^6-12x^5+22x^4+x^2+12x+4.$

14. $x^6-22x^4+34x^3+121x^2-374x+289.$

15. $a^2-ax+\frac{1}{4}x^2+8a-4x+16.$

16. $x^8+2x^7+x^6-4x^5-12x^4-8x^3+4x^2+16x+16.$

17. $16x^2-96x+216-\dfrac{216}{x}+\dfrac{81}{x^2}.$

18. $x^6-6x^4+15x^2-20+\dfrac{15}{x^2}-\dfrac{6}{x^4}+\dfrac{1}{x^6}.$

19. $x^4+2x^3(y+z)+x^2(y^2+z^2+4yz)+2xyz(y+z)+y^2z^2.$

20. $2x^2(y+z)^2+2y^2(z+x)^2+2z^2(x+y)^2+4xyz(x+y+z).$

CHAPTER XIX.

FRACTIONAL AND NEGATIVE INDICES.

178. We have hitherto supposed that an *index* was always a positive integer; and this is necessarily the case so long as we retain the definition of Art. 10; for, with that definition, such an expression as $a^{\frac{3}{2}}$, for example, has no meaning whatever.

It is however found convenient to extend the meaning of a^n, so as to include fractional and negative values of n.

Now it is essential that algebraical symbols should always obey the same laws whatever their values may be; we therefore do not begin by assigning any particular meaning to a^n, when n is not a positive integer, but we first impose the restriction that *the meaning of* a^n *must in all cases be such that the fundamental Index Law*, namely,

$$a^m \times a^n = a^{m+n},$$

shall always be true; and it will be found that the above restriction is of itself sufficient to define the meaning of a^n in all cases, so that there is no further freedom of choice.

For example, to find a meaning of $a^{\frac{1}{2}}$ consistent with the Index Law.

We must have

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a.$$

Thus $a^{\frac{1}{2}}$ must be such that its *square* is a, that is $a^{\frac{1}{2}}$ must mean $\sqrt{a}$.

Again, to find a meaning of a^{-1} consistent with the Index Law.

We must have $$a^3 \times a^{-1} = a^{3-1} = a^2;$$

$$\therefore \quad a^{-1} = a^2 \div a^3 = \frac{1}{a}.$$

Thus a^{-1} must mean $\frac{1}{a}$.

We now proceed to consider the most general cases.

179. To find the meaning of $a^{\frac{1}{n}}$, where n is any positive integer.

By the index law

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots \text{to } n \text{ factors}$$

$$= a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\ldots\text{to } n \text{ terms}} = a^{\frac{n}{n}} = a^1 = a.$$

Hence $a^{\frac{1}{n}}$ must be such that its nth power is a, that is

$$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

180. To find the meaning of $a^{\frac{m}{n}}$, where m and n are any positive integers.

By the index law

$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times \ldots \text{to } n \text{ factors}$$

$$= a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}+\ldots\text{to } n \text{ terms}} = a^{\frac{mn}{n}} = a^m.$$

Thus $a^{\frac{m}{n}}$ *is equal to the* n*th root of* a^m, that is

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

We have also

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots \text{to } m \text{ factors}$$

$$= a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\ldots\text{to } m \text{ terms}} = a^{\frac{m}{n}}.$$

Hence $a^{\frac{m}{n}}$ may be considered as the mth power of $a^{\frac{1}{n}}$, and by Art. 179,

$$a^{\frac{1}{n}} \text{ is } \sqrt[n]{a}.$$

Thus *we may consider that* $a^{\frac{m}{n}}$ *is the* n*th root of the* m*th power of* a, *or that it is the* m*th power of the* n*th root of* a; which we express by

$$a^{\frac{m}{n}} = \sqrt[n]{(a^m)} = (\sqrt[n]{a})^m.$$

Note. It should be remarked that it is not strictly true that $\sqrt[n]{(a^m)} = \{\sqrt[n]{(a)}\}^m$ unless by the nth root of a quantity is meant only the *arithmetical* root. For example $\sqrt[2]{(a^4)}$ has *two* values, namely $\pm a^2$, whereas $(\sqrt[2]{a})^4$ has only the value $+ a^2$.

Examples (i) $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$, (ii) $4^{\frac{3}{2}} = \sqrt[2]{4^3} = \sqrt[2]{64} = 8$,
(iii) $3^{\frac{5}{2}} = \sqrt[2]{3^5} = \sqrt[2]{243}$.

181. To find the meaning of a^0.

By the index law

$$a^0 \times a^m = a^{0+m} = a^m;$$

$$\therefore\ a^0 = a^m \div a^m = 1.$$

Thus, *whatever* a *may be*, $a^0 = 1$.

182. To find the meaning of a^{-m}, where m has any positive value.

By the index law

$$a^m \times a^{-m} = a^{m-m} = a^0,$$

and, by the preceding Article, $a^0 = 1$;

$$\therefore\ a^m \times a^{-m} = 1.$$

so that $$a^{-m} = \frac{1}{a^m}, \text{ and } a^m = \frac{1}{a^{-m}}.$$

Thus *we can change any quantity from the numerator to the denominator, or from the denominator to the numerator, of a fraction, provided we change the sign of its index.*

For example

$$2^{-2}=\frac{1}{2^2}=\frac{1}{4},$$

$$\frac{a}{b}=ab^{-1}=\frac{1}{a^{-1}b},$$

and

$$\frac{a^3b^2}{x^3y}=a^3b^2x^{-3}y^{-1}=\frac{1}{a^{-3}b^{-2}x^3y}.$$

183. We have in the preceding Articles found that in order that the fundamental Index Law, namely $a^m\times a^n=a^{m+n}$, may always be obeyed, a^m must have a definite meaning when n has any given positive or negative value. It can be proved* that, *with the meanings thus obtained*,

$$a^m\times a^n=a^{m+n},\quad (a^m)^n=a^{mn},\quad \text{and}\quad (ab)^n=a^nb^n,$$

are true for all values of m and n.

It therefore follows that we may deal with quantities containing fractional or negative indices in precisely the same way as if the indices were positive integers.

For example

$$(a^{\frac{2}{3}}b^{\frac{1}{2}})^2=a^{\frac{2}{3}\times 2}b^{\frac{1}{2}\times 2}=a^{\frac{4}{3}}b.$$

$$a^{\frac{2}{3}}\times a^{-\frac{1}{2}}=a^{\frac{2}{3}-\frac{1}{2}}=a^{\frac{1}{6}}=\sqrt[6]{a}.$$

$$a^{\frac{1}{2}}\times a^{\frac{1}{6}}\times a^{-\frac{2}{3}}=a^{\frac{1}{2}+\frac{1}{6}-\frac{2}{3}}=a^0=1.$$

$$(a^{-\frac{2}{3}})^6=a^{-\frac{2}{3}\times 6}=a^{-4}=\frac{1}{a^4}.$$

$$a^{-2}\div a^{-4}=a^{-2-(-4)}=a^2.$$

$$\sqrt[3]{(a^3b^{-3}c^4)}=\sqrt[3]{(a^3)}\,.\,\sqrt[3]{(b^{-3})}\,.\,\sqrt[3]{(c^4)}=a^{\frac{3}{3}}b^{-\frac{3}{3}}c^{\frac{4}{3}}=ab^{-1}c^{\frac{4}{3}}.$$

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* See *Treatise on Algebra*, Art. 168.

$a^{\frac{1}{n}} = \sqrt[n]{a}$

EXAMPLES. LVI.

Find the values of

1. $8^{\frac{2}{3}}$. 2. $4^{-\frac{3}{2}}$. 3. $16^{-\frac{1}{2}}$. 4. $(\frac{1}{25})^{-\frac{1}{2}}$.

5. $(\frac{4}{25})^{-\frac{3}{2}}$. 6. $(\frac{27}{64})^{-\frac{2}{3}}$. 7. $(27)^{-\frac{4}{3}}$. 8. $(100)^{-\frac{5}{2}}$.

9. $(\frac{1}{10000})^{-\frac{3}{4}}$. 10. $(\frac{27}{125})^{-\frac{4}{3}}$.

Simplify

11. $a^{\frac{2}{3}} \times a^{-\frac{1}{3}}$. 12. $a^{\frac{4}{3}} \times a^{\frac{2}{3}}$. 13. $a^{-\frac{2}{3}} \times a$.

14. $a \times a^{-\frac{2}{5}}$. 15. $a^{\frac{2}{3}}b^{-\frac{1}{3}} \times a^{-\frac{1}{3}}b^{\frac{2}{3}}$. 16. $(a^{\frac{1}{2}}b)^2$.

17. $(a^{\frac{3}{2}}b^{-\frac{1}{2}})^2$. 18. $(a^{\frac{1}{3}}b^{-\frac{2}{3}})^3$. 19. $(a^{-\frac{2}{3}}b^{\frac{1}{6}})^{\frac{3}{2}}$.

20. $(a^4b^{-\frac{2}{3}})^{\frac{3}{2}}$. 21. $\{(a^{-2})^2\}^{\frac{3}{4}}$.

22. $\{(a^{-\frac{1}{3}})^2\}^{-\frac{3}{2}}$. 23. $\{(a^{-\frac{5}{6}})^3\}^{-\frac{1}{5}}$.

24. $a^{\frac{1}{3}} \times a^{-\frac{3}{4}} \times a^{\frac{5}{12}}$. 25. $a^{\frac{2}{3}} \times a^{-\frac{3}{4}} \times (a^2)^{-\frac{1}{6}}(a^{\frac{1}{12}})^5$.

26. $x^{p+q} \times x^{p-q} \div x^{2p}$. 27. $(x^{q-r})^p \times (x^{r-p})^q \times (x^{p-q})^r$.

28. $\dfrac{x^{\frac{1}{2}}y^{\frac{2}{3}}}{x^{\frac{1}{3}}y^{\frac{1}{2}}} \div \dfrac{x^{\frac{1}{6}}}{y^{\frac{1}{6}}}$. 29. $\dfrac{(xy)^{x+y}}{x^yy^x}$.

30. $\dfrac{(xyz)^{x+y+z}}{x^{y+z}y^{z+x}z^{x+y}}$. 31. $\dfrac{(a^{\frac{1}{2}})^{\frac{1}{3}}(b^{\frac{1}{2}})^6}{(b^{\frac{1}{4}})^3a^{\frac{1}{3}}} \div \dfrac{(a^{\frac{2}{3}})^{\frac{1}{4}}b^{\frac{1}{12}}}{(b^6)^{\frac{1}{7}}}$.

32. $a^{\frac{1}{2}}y^{\frac{1}{3}} \times \left(\dfrac{y^{\frac{1}{4}}}{x^{\frac{1}{6}}}\right)^2 \div \dfrac{y^{-\frac{1}{4}}}{x^{\frac{1}{4}}}$. 33. $x^{3p+q} \times x^{p-4r} \times (x^2)^{q-2r} \div x^{4p-8r}$.

34. $\left\{x^{\frac{p-q}{pq}}\right\}^r\left\{x^{\frac{q-r}{qr}}\right\}^p\left\{x^{\frac{r-p}{rp}}\right\}^q \div x^{\frac{p-q}{r}+\frac{q-r}{p}+\frac{r-p}{q}}$.

35. $\left[\left\{x^{\frac{p-q}{r}}\right\}^{\frac{q-r}{p}}\right]^{\frac{r-p}{q}} \times x^{\frac{p-q}{r}+\frac{q-r}{p}+\frac{r-p}{q}}$.

Express with fractional indices and simplify

36. $\sqrt[3]{a} \times \sqrt[2]{a} \times \sqrt[6]{a}$. 37. $\sqrt[3]{(a^2y)} \times \sqrt[3]{(ay^2)}$.

38. $\sqrt[3]{(a^2x^5)} \times \sqrt{(a^3x)}$. 39. $\sqrt[7]{a^3} \times \sqrt[6]{a^7} \times a^{-\frac{1}{2}} \div a^{\frac{2}{21}}$.

40. $\sqrt[3]{a^7} \times \sqrt[5]{a^7} \times a^{-\frac{1}{3}} \div a^{\frac{4}{5}}$. 41. $\sqrt[4]{a^2\sqrt[3]{\{a^6\sqrt{(a^{-23})}\}}}$.

42. $\sqrt[5]{}(a^2b^{10}c^5)\div\sqrt[3]{}(ab^6c^3)$. **43.** $\sqrt{}(a^3b^{-2})\div\sqrt[3]{}(a^{-4}b^5)$.

44. $\sqrt[3]{}(a^6b^{\frac{3}{2}}c^{\frac{1}{2}})\times b^{-\frac{9}{2}}\times(c^{\frac{1}{3}}a^4)^{-\frac{1}{2}}$.

Express the following quantities with radical signs and positive indices.

45. $a^{\frac{1}{2}}-a^{-2}$. **46.** $a^{-4}b^{-\frac{1}{4}}$. **47.** $a^{\frac{2}{3}}b^{-1}-a^{-\frac{2}{3}}b^{-\frac{1}{2}}$.

48. $a^{-3}b^{-\frac{1}{2}}+3^{-2}a^{\frac{2}{3}}b^{-\frac{2}{3}}$.

184. In the following examples the foregoing principles are applied, but the results of the operations of addition &c. which have to be performed on the indices are given without any of the intermediate steps.

Ex. 1. Multiply $a^{\frac{1}{3}}+1+a^{-\frac{1}{3}}$ by $a^{\frac{1}{3}}-1+a^{-\frac{1}{3}}$.

$$\begin{array}{lllll}
a^{\frac{1}{3}}+1 & +a^{-\frac{1}{3}} & & & \\
a^{\frac{1}{3}}-1 & +a^{-\frac{1}{3}} & & & \\
\hline
a^{\frac{2}{3}}+a^{\frac{1}{3}} & +1 & & & \\
 & -a^{\frac{1}{3}} & -1 & -a^{-\frac{1}{3}} & \\
 & & 1 & +a^{-\frac{1}{3}} & +a^{-\frac{2}{3}} \\
\hline
a^{\frac{2}{3}} & & +1 & & +a^{-\frac{2}{3}}.
\end{array}$$

Ex. 2. Divide $a+b+c-3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}$.

$$+b^{\frac{1}{3}}+c^{\frac{1}{3}}\,)\,a-3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}+b+c\,(\,a^{\frac{2}{3}}-a^{\frac{1}{3}}(b^{\frac{1}{3}}+c^{\frac{1}{3}})+b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}}$$

$$\begin{array}{l}
\underline{a+\ a^{\frac{2}{3}}(b^{\frac{1}{3}}+c^{\frac{1}{3}})} \\
-\ a^{\frac{2}{3}}(b^{\frac{1}{3}}+c^{\frac{1}{3}})-3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}+b+c \\
\underline{-\ a^{\frac{2}{3}}(b^{\frac{1}{3}}+c^{\frac{1}{3}})-\ a^{\frac{1}{3}}(b^{\frac{1}{3}}+c^{\frac{1}{3}})^2} \\
\qquad\qquad\qquad a^{\frac{1}{3}}(b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}})+b+c \\
\qquad\qquad\qquad \underline{a^{\frac{1}{3}}(b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}})+(b^{\frac{1}{3}}+c^{\frac{1}{3}})(b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}})}
\end{array}$$

Ex. 3. Find the square root of $x^{\frac{5}{3}}-4x^{\frac{4}{3}}+2x^{\frac{7}{6}}+4x-4x^{\frac{5}{6}}+x^{\frac{2}{3}}$.

$$\begin{array}{r l}
 & x^{\frac{5}{3}}-4x^{\frac{4}{3}}+2x^{\frac{7}{6}}+4x-4x^{\frac{5}{6}}+x^{\frac{2}{3}}\ (x^{\frac{5}{6}}-2x^{\frac{1}{2}}+x^{\frac{1}{3}} \\
(x^{\frac{5}{6}})^2= & x^{\frac{5}{3}} \\
\hline
(x^{\frac{5}{6}}-2x^{\frac{1}{2}})^2= & x^{\frac{5}{3}}-4x^{\frac{4}{3}}+4x \\
\hline
(x^{\frac{5}{6}}-2x^{\frac{1}{2}}+x^{\frac{1}{3}})^2= & x^{\frac{5}{6}}-4x^{\frac{4}{3}}+2x^{\frac{7}{6}}+4x-4x^{\frac{5}{6}}+x^{\frac{2}{3}}.
\end{array}$$

EXAMPLES LVII.

Multiply

1. $x^{\frac{1}{2}}+y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$. **2.** $x^{\frac{3}{2}}+y^{\frac{3}{2}}$ by $x^{\frac{3}{2}}-y^{\frac{3}{2}}$.

3. $x^{\frac{1}{4}}+1$ by $x^{\frac{1}{4}}-1$. **4.** $x^{\frac{1}{3}}+y^{\frac{1}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$.

5. $x^{\frac{2}{3}}+x^{\frac{1}{3}}+1$ by $x^{\frac{1}{3}}-1$. **6.** $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}$.

7. $a^{\frac{4}{3}}+a^{\frac{2}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}}$ by $a^{\frac{2}{3}}-b^{\frac{2}{3}}$.

8. $a^{\frac{4}{3}}+a^{\frac{2}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}}$ by $a^{\frac{4}{3}}-a^{\frac{2}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}}$.

9. $x^{\frac{5}{4}}-x^{\frac{3}{4}}+x^{\frac{1}{4}}-x^{-\frac{1}{4}}$ by $x^{\frac{3}{4}}+x^{\frac{1}{4}}$.

10. $x^{\frac{7}{4}}-x^{\frac{3}{2}}+x^{\frac{5}{4}}-x$ by $x^{\frac{3}{4}}+x^{\frac{1}{2}}$.

11. $x^n+x^{\frac{n}{2}}+1$ by $x^{-n}+x^{-\frac{n}{2}}+1$.

12. $x^{2n}+x^ny^n+y^{2n}$ by $x^{-2n}+x^{-n}y^{-n}+y^{-2n}$.

13. $a^{\frac{2}{3}}+b^{\frac{2}{3}}+c^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}-c^{\frac{1}{3}}a^{\frac{1}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}$.

14. $\frac{1}{8}a^{\frac{3}{2}}-\frac{1}{12}ab^{\frac{1}{2}}+\frac{1}{18}a^{\frac{1}{2}}b-\frac{1}{27}b^{\frac{3}{2}}$ by $\frac{1}{2}a^{\frac{1}{2}}+\frac{1}{3}b^{\frac{1}{2}}$.

15. $81x^{\frac{4}{5}}-27x^{\frac{3}{5}}y^{\frac{1}{3}}+9x^{\frac{2}{5}}y^{\frac{2}{3}}-3x^{\frac{1}{5}}y+y^{\frac{4}{3}}$ by $3x^{\frac{1}{5}}+y^{\frac{1}{3}}$.

Divide

16. $a-b$ by $a^{\frac{1}{2}}-b^{\frac{1}{2}}$. **17.** x^3-y^3 by $x^{\frac{3}{2}}-y^{\frac{3}{2}}$.

18. $x^{\frac{3n}{2}}-y^{\frac{3n}{2}}$ by $x^{\frac{n}{2}}-y^{\frac{n}{2}}$. **19.** x^2+y^2 by $x^{\frac{2}{3}}+y^{\frac{2}{3}}$.

20. $x-243y^{\frac{5}{3}}$ by $x^{\frac{1}{5}}-3y^{\frac{1}{3}}$. **21.** $y^4+b^2y^2+b^4$ by $y-b^{\frac{1}{2}}y^{\frac{1}{2}}+b$.

22. $x^{\frac{3}{4}}y^{-\frac{3}{4}}+2+x^{-\frac{3}{4}}y^{\frac{3}{4}}$ by $x^{\frac{1}{4}}y^{-\frac{1}{4}}-1+x^{-\frac{1}{4}}y^{\frac{1}{4}}$.

23. $a^{\frac{5}{2}}+a^2b^{\frac{1}{3}}-a^{\frac{3}{2}}b^{\frac{2}{3}}-ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}$ by $a^{\frac{1}{2}}+b^{\frac{1}{3}}$.

24. $a^{\frac{4}{3}}-2+a^{-\frac{4}{3}}$ by $a^{\frac{2}{3}}-a^{-\frac{2}{3}}$.

25. $(x-x^{-1})-2(x^{\frac{1}{6}}-x^{-\frac{1}{6}})+2(x^{\frac{5}{6}}-x^{-\frac{5}{6}})$ by $x^{\frac{1}{3}}-x^{-\frac{1}{3}}$.

26. $x^{\frac{7}{3}}y^{-\frac{14}{5}}+y^{\frac{14}{5}}x^{-\frac{7}{3}}$ by $x^{\frac{1}{3}}y^{-\frac{2}{5}}+y^{\frac{2}{5}}x^{-\frac{1}{3}}$.

27. Simplify $(ab^{-3}c^5)^{\frac{1}{2}}\times(a^7b^4c^{-1})^{\frac{1}{3}}\times(a^{-5}bc^2)^{\frac{1}{6}}$.

28. Simplify $(a-4b^{\frac{2}{3}})(a+2a^{\frac{1}{2}}b^{\frac{1}{3}}+4b^{\frac{2}{3}})(a-2a^{\frac{1}{2}}b^{\frac{1}{3}}+4b^{\frac{2}{3}})$.

29. Shew that $1-\dfrac{a^2+b^2-a^{-2}-b^{-2}}{a^2b^2-a^{-2}b^{-2}}=\dfrac{(a-a^{-1})(b-b^{-1})}{ab+a^{-1}b^{-1}}$.

30. Shew that $\dfrac{x}{x^{\frac{1}{3}}-1}-\dfrac{x^{\frac{2}{3}}}{x^{\frac{1}{3}}+1}-\dfrac{1}{x^{\frac{1}{3}}-1}+\dfrac{1}{x^{\frac{1}{3}}+1}=2+x^{\frac{2}{3}}$.

31. Multiply $4x^2-5x-4-7x^{-1}+6x^{-2}$ by $3x-4+2x^{-1}$, and divide the product by $3x-10+10x^{-1}-4x^{-2}$.

32. Simplify $\dfrac{1}{1-x^{\frac{1}{4}}}+\dfrac{1}{1+x^{\frac{1}{4}}}+\dfrac{2}{1+x^{\frac{1}{2}}}+\dfrac{4}{1+x}$.

33. Multiply $(a+b)^{\frac{2}{3}}+(a^2-b^2)^{\frac{1}{3}}+(a-b)^{\frac{2}{3}}$ by $(a+b)^{\frac{1}{3}}-(a-b)^{\frac{1}{3}}$.

34. Write down the square roots of

(i) $x^{\frac{2}{3}}+2x^{\frac{1}{3}}+1$, (ii) $4x^{\frac{2}{3}}-4x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$,

(iii) $ax^{\frac{2}{3}}-2a^{\frac{5}{6}}x^{\frac{5}{6}}+a^{\frac{2}{3}}x$.

35. Find the square root of $4x^2a^{-2}-12xa^{-1}+25-24x^{-1}a+16x^{-2}a^2$.

36. Find the square root of $25x^2y^{-2}+\frac{1}{4}y^2x^{-2}-20xy^{-1}-2yx^{-1}+9$.

37. Find the square root of

$$x^{\frac{8}{5}}-2a^{-\frac{3}{5}}x^{\frac{11}{5}}+2a^{\frac{4}{5}}x^{\frac{4}{5}}+a^{-\frac{6}{5}}x^{\frac{14}{5}}-2a^{\frac{1}{5}}x^{\frac{7}{5}}+a^{\frac{8}{5}}.$$

38. Find the square root of $x^{\frac{2}{3}}-4x^{\frac{5}{6}}+4x+2x^{\frac{7}{6}}-4x^{\frac{4}{3}}+x^{\frac{5}{3}}$.

39. Solve the equations

(i) $x^{\frac{3}{2}}-2x^{\frac{3}{4}}+1=0$.

(ii) $x-8x^{\frac{1}{2}}+12=0$.

(iii) $x^{\frac{3}{2}}-26x^{\frac{3}{4}}-27=0$.

(iv) $x^{\frac{1}{3}}+3x^{-\frac{1}{3}}=4$.

(v) $4x^{\frac{1}{5}}-3x^{-\frac{1}{5}}=4$.

CHAPTER XX.

SURDS.

185. **Definitions.** A *surd* is a root of an arithmetical number which can only be found approximately.

Thus $\sqrt{2}$ and $\sqrt[3]{4}$ are surds.

An algebraical expression such as $\sqrt{a}$ is also often called a surd, although a *may* have such a value that $\sqrt{a}$ is not in reality a surd.

Surds are said to be of the same **order** when the same root is required to be taken. Thus $\sqrt{2}$ and $6^{\frac{1}{2}}$ are called surds of the *second* order, or **quadratic** surds; also $\sqrt[3]{4}$ and $5^{\frac{2}{3}}$ are surds of the *third* order, or **cubic** surds; and $\sqrt[n]{5}$ is a surd of the nth order.

Two surds are said to be **similar** when they can be reduced so as to have the same irrational factors.

Thus $2\sqrt{2}$ and $5\sqrt{2}$ are similar surds.

The rules for operations with surds follow at once from the principles already established.

Note. It should be remarked that when a root symbol is placed before an *arithmetical* number it denotes only the *arithmetical root*, but when a root symbol is placed before an algebraical expression it denotes any *one* of the roots. Thus $\sqrt{a}$ has *two* values, but $\sqrt{2}$ is only supposed to denote the arithmetical root, unless it is actually written $\pm\sqrt{2}$.

186. Any rational quantity can be written in the form of a surd.

For example

$$a=\sqrt[2]{a^2}=\sqrt[3]{a^3}=\sqrt[n]{a^n},$$

and $$2=\sqrt[2]{2^2}=\sqrt[3]{2^3}=\sqrt[n]{2^n}.$$

Again $$\sqrt[3]{2}=\sqrt[3]{\{\sqrt{2^2}\}}=\sqrt[6]{2^2}.$$

Also, since $\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}$, we have

$$3\sqrt{2}=\sqrt{3^2}\times\sqrt{2}=\sqrt{(3^2\times 2)}=\sqrt{18},$$

$$2\sqrt[3]{5}=\sqrt[3]{2^3}\times\sqrt[3]{5}=\sqrt[3]{(2^3\times 5)}=\sqrt[3]{40},$$

and $$a\sqrt[n]{b}=\sqrt[n]{a^n}\times\sqrt[n]{b}=\sqrt[n]{a^nb}.$$

Conversely, we have

$$\sqrt{18}=\sqrt{(9\times 2)}=\sqrt{9}\times\sqrt{2}=3\sqrt{2},$$

$$\sqrt[3]{135}+\sqrt[3]{40}=\sqrt[3]{(3^3\times 5)}+\sqrt[3]{(2^3\times 5)}=3\sqrt[3]{5}+2\sqrt{5}=5\sqrt[3]{5},$$

and $$\sqrt{a^3}+\sqrt{ab^2}=a\sqrt{a}+b\sqrt{a}=(a+b)\sqrt{a}.$$

187. *Any two surds can be reduced to surds of the same order.*

This follows at once from the fact that $\sqrt[n]{a}=\sqrt[nm]{a^m}$.

For example, to reduce $\sqrt{2}$ and $\sqrt[3]{5}$ to surds of the same order.

We have $\sqrt[2]{2}=\sqrt[2]{\{\sqrt[3]{2^3}\}}=\sqrt[6]{2^3}$, and $\sqrt[3]{5}=\sqrt[3]{\{\sqrt[2]{5^2}\}}=\sqrt[6]{5^2}$; thus the equivalent surds of the same order are $\sqrt[6]{8}$ and $\sqrt[6]{25}$.

Again, to reduce $\sqrt[n]{a}$ and $\sqrt[m]{b}$ to equivalent surds of the same order.

We have $\sqrt[n]{a}=\sqrt[n]{\{\sqrt[m]{a^m}\}}=\sqrt[mn]{a^m}$, and $\sqrt[m]{b}=\sqrt[m]{\{\sqrt[n]{b^n}\}}=\sqrt[mn]{b^n}$. Hence the required surds are $\sqrt[nm]{a^m}$ and $\sqrt[nm]{b^n}$.

Ex. 1. Reduce $\sqrt[4]{3}$ and $\sqrt[6]{5}$ to surds of the same order.

The L.C.M. of 4 and 6 is 12. Hence we have

$$\sqrt[4]{3}=\sqrt[4]{\{\sqrt[3]{3^3}\}}=\sqrt[12]{27},\text{ and }\sqrt[6]{5}=\sqrt[6]{\{\sqrt[2]{5^2}\}}=\sqrt[12]{25}.$$

Thus $\sqrt[12]{27}$ and $\sqrt[12]{25}$ are the required surds.

Ex. 2. Which is the greater $\sqrt[3]{14}$ or $\sqrt[2]{6}$? We must reduce the surds to equivalent surds of the same order.

Thus $\sqrt[3]{14}=\sqrt[6]{14^2}=\sqrt[6]{196}$,

and $\sqrt[2]{6}=\sqrt[6]{6^3}=\sqrt[6]{216}$.

It is now obvious that $\sqrt[2]{6}$ is greater than $\sqrt[3]{14}$.

188. The product of two surds of the *same order* can be written down at once from the formula

$$\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}.$$

When surds are of different orders their product cannot be simplified until they are reduced to equivalent surds of the same order.

Ex. 1. Multiply $\sqrt{5}$ by $\sqrt{20}$.

We have $\sqrt{5}\times\sqrt{20}=\sqrt{(5\times 20)}=\sqrt{100}=10$.

Ex. 2. Multiply $\sqrt{2}$ by $\sqrt[3]{3}$.

$$\sqrt{2}\times\sqrt[3]{3}=\sqrt[6]{2^3}\times\sqrt[6]{3^2}=\sqrt[6]{(2^3\times 3^2)}=\sqrt[6]{72}.$$

Ex. 3. Divide $\sqrt[3]{2}$ by $\sqrt{6}$.

$$\sqrt[3]{2}\div\sqrt{6}=\sqrt[6]{2^2}\div\sqrt[6]{6^3}=\sqrt[6]{(2^2\div 6^3)}=\sqrt[6]{\tfrac{1}{54}}.$$

Ex. 4. Multiply $4\sqrt{3}+4\sqrt{2}$ by $2\sqrt{3}-2\sqrt{2}$.

The process is as under:

$$\begin{array}{llll}
4\sqrt{3} & +4\sqrt{2} & & \\
2\sqrt{3} & -2\sqrt{2} & & \\
\hline
8\times 3 & +8\sqrt{6} & & \\
 & -8\sqrt{6} & -8\times 2 & \\
\hline
24 & & -16 & =8.
\end{array}$$

EXAMPLES. LVIII.

Simplify.

1. $\sqrt{27}+\sqrt{48}$.
2. $\sqrt{50}+\sqrt{98}$.
3. $\sqrt{45}+2\sqrt{125}$.
4. $2\sqrt{180}-\sqrt{405}$.
5. $2\sqrt{28}-\sqrt{63}$.
6. $5\sqrt{208}-3\sqrt{325}$.
7. $3\sqrt[3]{5}+\sqrt[3]{625}$.
8. $3\sqrt[3]{72}-2\sqrt[3]{243}$.
9. $4\sqrt[3]{448}-15\sqrt[3]{7}$.
10. $\sqrt{512}-\sqrt{50}-\sqrt{98}$.
11. $3\sqrt{12}-\sqrt{27}+2\sqrt{75}$.

12. $\sqrt{147}-2\sqrt{27}-\sqrt{3}$.

13. $5\sqrt{\frac{1}{2}}-\sqrt{8}+\frac{1}{\sqrt{18}}$.

14. $\sqrt{15}\times\sqrt{60}$.

15. $\sqrt{12}\times\sqrt{24}$.

16. $2\sqrt{\frac{2}{3}}\times 3\sqrt{\frac{3}{2}}$.

17. $\sqrt[3]{6}\times\sqrt[3]{36}$.

18. $\sqrt{12}\times\sqrt{27}\times\sqrt{75}$.

19. $\sqrt[3]{16}\times\sqrt[3]{6}\times\sqrt[3]{9}$.

20. $\sqrt[3]{12}\times\sqrt[3]{75}\times\sqrt[3]{30}$.

21. $\sqrt[4]{6}\times\sqrt[4]{12}\times\sqrt[4]{18}$.

22. $\sqrt{10}\times\sqrt[3]{200}$.

23. $\sqrt[3]{4}\times\sqrt[2]{8}$.

24. $\sqrt{18}\div\sqrt{50}$.

25. $\sqrt{63}\div\sqrt{112}$.

26. $\sqrt{20}\times\sqrt{96}\div\sqrt{30}$.

27. $\sqrt[3]{147}\div\sqrt[3]{35}\times\sqrt[3]{735}$.

28. Which is the greater $\sqrt{3}$ or $\sqrt[3]{(5\frac{1}{5})}$?

29. Arrange in order of magnitude $\sqrt{50}$, $\sqrt[3]{344}$ and $\sqrt[4]{2402}$.

30. Multiply $\sqrt{6}-\sqrt{3}$ by $\sqrt{6}+\sqrt{3}$.

31. Multiply $2\sqrt{5}+3\sqrt{3}$ by $3\sqrt{5}-4\sqrt{3}$.

32. Multiply $\sqrt{2}+\sqrt{3}+\sqrt{6}$ by $2\sqrt{2}+3\sqrt{3}+\sqrt{6}$.

33. Multiply $1+\sqrt[3]{3}+\sqrt[3]{5}$ by $1+\sqrt[3]{3}-\sqrt[3]{5}$.

34. Find the square of $\sqrt{2}+\sqrt{3}+\sqrt{5}$.

35. Find the continued product of $\sqrt{2}+\sqrt{3}+\sqrt{5}$, $-\sqrt{2}+\sqrt{3}+\sqrt{5}$, $\sqrt{2}-\sqrt{3}+\sqrt{5}$ and $\sqrt{2}+\sqrt{3}-\sqrt{5}$.

189. When surds occur in the denominators of fractions they can be got rid of, and the denominators are then said to have been rationalised.

The following examples will sufficiently illustrate the process :

$$\frac{3}{\sqrt{5}}=\frac{3\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}}=\frac{3}{5}\sqrt{5},$$

$$\frac{3\sqrt{2}}{\sqrt{7}}=\frac{3\sqrt{2}\times\sqrt{7}}{\sqrt{7}\times\sqrt{7}}=\frac{3}{7}\sqrt{14},$$

$$\frac{2+\sqrt{5}}{\sqrt{5}-1}=\frac{(2+\sqrt{5})(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)}=\frac{2\sqrt{5}+5+2+\sqrt{5}}{5-1}$$

$$=\tfrac{1}{4}(7+3\sqrt{5}),$$

$$\frac{a-\sqrt{b}}{a+\sqrt{b}}=\frac{(a-\sqrt{b})(a-\sqrt{b})}{(a+\sqrt{b})(a-\sqrt{b})}=\frac{a^2-2a\sqrt{b}+b}{a^2-b}.$$

It is important to notice that $a \pm \sqrt{b}$ is made rational by multiplying by $a \mp \sqrt{b}$; also that $\sqrt{a} \pm \sqrt{b}$ is made rational by multiplying by $\sqrt{a} \mp \sqrt{b}$.

When the denominator of a fraction is rationalised its numerical value can be more easily found.

190. The following is an important proposition.

If $a + \sqrt{b} = \alpha + \sqrt{\beta}$, *where* a *and* α *are rational, and* $\sqrt{b}$ *and* $\sqrt{\beta}$ *are irrational; then will* $a = \alpha$, *and* $b = \beta$.

For we have $\quad a - \alpha + \sqrt{b} = \sqrt{\beta}$.

Square both sides; then, after transformation, we have

$$2(a-\alpha)\sqrt{b} = \beta - b - (a-\alpha)^2.$$

Hence, *unless the coefficient of* $\sqrt{b}$ *is zero*, we must have an irrational and a rational quantity equal to one another, and this is impossible.

The coefficient of $\sqrt{b}$ in the last equation must therefore be zero; hence $a = \alpha$. And when $a = \alpha$, we have from the given relation, $\sqrt{b} = \sqrt{\beta}$, or $b = \beta$.

Hence, *if the sum (or difference) of a rational quantity and a quadratic surd be equal to the sum (or difference) of another rational quantity and a quadratic surd, the two rational quantities must be equal to one another, as also the two irrational quantities.*

> **Note.** It should be noticed that when $a + \sqrt{b} = \alpha + \sqrt{\beta}$, we can only conclude that $a = \alpha$ and $b = \beta$ *provided that* $\sqrt{b}$ *and* $\sqrt{\beta}$ *are really irrational.* We cannot for example, from the relation $3 + \sqrt{4} = 2 + \sqrt{9}$, conclude that $3 = 2$ and $4 = 9$.

191. The square root of a binomial expression which is the sum of a rational quantity and a quadratic surd can sometimes be found in a simple form. The process is as under.

To find $\sqrt{(a+\sqrt{b})}$, where $\sqrt{b}$ is a surd.

Let $$\sqrt{(a+\sqrt{b})}=\sqrt{\alpha}+\sqrt{\beta}\text{..................(i).}$$

Square both sides; then

$$a+\sqrt{b}=\alpha+\beta+2\sqrt{\alpha\beta}.$$

Now, since $\sqrt{b}$ is a surd, we can equate the rational and irrational parts [Art. 190]; hence

$$\left.\begin{aligned}a&=\alpha+\beta\\ b&=4\alpha\beta\end{aligned}\right\}\text{.....................(ii).}$$

Hence [Art. 150] α and β are the roots of the equation

$$x^2-ax+\frac{b}{4}=0\,;$$

and these roots are

$$\left\{\frac{a+\sqrt{(a^2-b)}}{2}\right\}\text{ and }\left\{\frac{a-\sqrt{(a^2-b)}}{2}\right\}.$$

Thus $$\sqrt{(a+\sqrt{b})}=\sqrt{\left\{\frac{a+\sqrt{(a^2-b)}}{2}\right\}}$$
$$+\sqrt{\left\{\frac{a-\sqrt{(a^2-b)}}{2}\right\}}\text{..............(iii).}$$

It is clear, that unless $\sqrt{(a^2-b)}$ is rational, the right side of (iii) is much more complicated than the left. Thus the above process fails entirely unless a^2-b is a square number; and as this condition will not often be satisfied, the process has no great practical utility.

From (ii) we see that we have to find two numbers whose sum is a and whose product is $\frac{b}{4}$; and if two *rational* numbers satisfy these conditions, they can generally be found at once by inspection.

Ex. 1. Find $\sqrt{(15+2\sqrt{56})}$.

Let $$\sqrt{(15+2\sqrt{56})}=\sqrt{\alpha}+\sqrt{\beta}.$$

Square both sides; then

$$15+2\sqrt{56}=\alpha+\beta+2\sqrt{\alpha\beta}.$$

Equalizing the rational and irrational terms, we have

$$\alpha+\beta=15,$$
$$\alpha\beta=56.$$

The numbers which satisfy these relations are obviously 7 and 8.

Hence $$\sqrt{(15+2\sqrt{56})}=\sqrt{7}+\sqrt{8}.$$

Ex. 2. Find $\sqrt{(6-\sqrt{35})}$.

Let $$\sqrt{(6-\sqrt{35})}=\sqrt{\alpha}-\sqrt{\beta}.$$

Square both sides; then

$$6-\sqrt{35}=\alpha+\beta-2\sqrt{\alpha\beta}.$$

Hence, equating the rational and irrational terms, we have

$$\alpha+\beta=6,$$
$$\alpha\beta=\tfrac{35}{4}.$$

By inspection, or by solving the equations for α and β, we find that $$\alpha=\tfrac{7}{2},\ \beta=\tfrac{5}{2}.$$

Thus $$\sqrt{(6-\sqrt{35})}=\sqrt{\tfrac{7}{2}}-\sqrt{\tfrac{5}{2}}.$$

EXAMPLES. LIX.

Rationalise the denominators of

1. $\dfrac{3}{\sqrt{7}}$.
2. $\dfrac{2}{\sqrt{5}}$.
3. $\dfrac{3\sqrt{2}}{2\sqrt{3}}$.
4. $\dfrac{\sqrt{6}}{\sqrt{5}}$.
5. $\dfrac{2}{1+\sqrt{2}}$.
6. $\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$.
7. $\dfrac{2\sqrt{5}}{\sqrt{5}+\sqrt{3}}$.
8. $\dfrac{15+14\sqrt{3}}{15-2\sqrt{3}}$.
9. $\dfrac{\sqrt{5}+3\sqrt{3}}{2\sqrt{5}-\sqrt{3}}$.
10. $\dfrac{\sqrt{6}-3\sqrt{12}}{2\sqrt{6}+\sqrt{12}}$.
11. $\dfrac{1}{1+\sqrt{2}+\sqrt{3}}$.
12. $\dfrac{3}{2+\sqrt{3}+\sqrt{5}}$.
13. $\dfrac{1}{\sqrt[3]{2}-1}+\dfrac{3}{\sqrt[3]{2}+1}$.
14. $\dfrac{4}{\sqrt[3]{9}-1}+\dfrac{5}{\sqrt[3]{9}+1}$.

15. Simplify $\dfrac{1}{(2-\sqrt{3})^2}+\dfrac{1}{(2+\sqrt{3})^2}$.

16. Simplify $\dfrac{1}{(2-\sqrt{3})^3}+\dfrac{1}{(2+\sqrt{3})^3}$.

17. Simplify $\dfrac{(3+\sqrt{2})(5-\sqrt{2})}{(3-\sqrt{2})(5+\sqrt{2})}$.

18. Simplify $\dfrac{2\sqrt{15}-3\sqrt{5}+2\sqrt{2}}{\sqrt{15}+\sqrt{2}}$.

19. Simplify $\dfrac{(7-2\sqrt{5})(5+\sqrt{7})(31+13\sqrt{5})}{(6-2\sqrt{7})(3+\sqrt{5})(11+4\sqrt{7})}$.

20. Simplify $\left(\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right)^2+\left(\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}\right)^2$.

21. Shew that $\dfrac{1+\sqrt{3}+\sqrt{6}}{1+\sqrt{2}-\sqrt{3}}=2+\sqrt{2}+\sqrt{3}+\sqrt{6}$.

22. Simplify

$$\frac{1}{1+\sqrt{2}+\sqrt{3}}+\frac{1}{-1+\sqrt{2}+\sqrt{3}}+\frac{1}{1-\sqrt{2}+\sqrt{3}}+\frac{1}{1+\sqrt{2}-\sqrt{3}}.$$

23. Simplify $\dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\dfrac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}$.

24. Shew that

$$\left(\frac{2-\sqrt{3}}{2}\right)^{-\frac{3}{2}}-\left(\frac{2+\sqrt{3}}{2}\right)^{\frac{3}{2}}=7\left(\frac{\sqrt{3}+1}{2}\right)^3.$$

25. Simplify $\dfrac{2\sqrt{(a+b)}+3\sqrt{(a-b)}}{2\sqrt{(a+b)}-\sqrt{(a-b)}}$.

26. Find the value of $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$ to three places of decimals.

27. Find the value of $\dfrac{\sqrt{3}}{2-\sqrt{3}}+\dfrac{2-\sqrt{2}}{2+\sqrt{2}}$ to three places of decimals.

Find the square roots of

28. $6+\sqrt{20}$.
29. $16+6\sqrt{7}$.
30. $12-6\sqrt{3}$.
31. $28-5\sqrt{12}$.
32. $101-28\sqrt{13}$.
33. $117+36\sqrt{10}$.
34. $280+56\sqrt{21}$.
35. $4\frac{1}{2}+2\sqrt{2}$.

Simplify

36. $3\sqrt{5}-\sqrt{2}+\sqrt{(7+2\sqrt{10})}$.

37. $6-4\sqrt{3}+\sqrt{(16-8\sqrt{3})}$.

Find the square roots of

38. $11+2(1+\sqrt{5})(1+\sqrt{7})$.

39. $2x+2\sqrt{(x^2-1)}$.

40. $2a+2\sqrt{(a^2-x^2)}$.

41. $3x-1+2\sqrt{(2x^2+x-6)}$.

Simplify

42. $\dfrac{1}{\sqrt{(16+6\sqrt{7})}}$.

43. $\dfrac{1}{\sqrt{(15+2\sqrt{56})}}$.

44. $\sqrt{(7+2\sqrt{10})}+\sqrt{(7-2\sqrt{10})}$.

45. $\dfrac{\sqrt{(3+2\sqrt{2})}-\sqrt{2}}{\sqrt{2}+\sqrt{(3-2\sqrt{2})}}$.

46. $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{(2+\sqrt{3})}}-\dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{(2-\sqrt{3})}}$.

47. Shew that $\dfrac{\sqrt{2}+\sqrt{45}}{\sqrt{2}+\sqrt{(7-2\sqrt{10})}}=3{\cdot}632\ldots$

48. Find the value of x^2-4x+5 when $x=2+\sqrt{5}$.

49. Shew that, if $x^2=x+1$, then $x^3=2x+1$ and $x^5=5x+3$.

50. Shew that, if $x^2=3x+5$, then $x^3=14x+15$ and $x^4=57x+70$.

$a : b$ - $3 : 4$

CHAPTER XXI.

RATIO. PROPORTION. VARIATION.

192. **Definitions.** The relative magnitude of two quantities, measured by the number of times which the one contains the other, is called their **ratio.**

Concrete quantities of different kinds can have no ratio to one another: we cannot, for example, compare with respect to magnitude miles and tons, or shillings and weeks.

The ratio of a to b is expressed by the notation $a : b$; and a is called the first **term**, and b the second **term** of the ratio.

Sometimes the first and second terms of a ratio are called respectively the **antecedent** and the **consequent.**

It is clear that a ratio is greater, equal, or less than unity, according as its first term is greater, equal, or less than the second.

A ratio which is greater than unity is sometimes called a ratio of **greater inequality**, and a ratio which is less than unity is similarly called a ratio of **less inequality**.

193. Magnitudes must always be expressed by means of *numbers;* and the number of times which one number is

contained in another is found by dividing the one by the other. Hence

$$a : b \text{ is equal to } \frac{a}{b}.$$

Thus ratios can be expressed as *fractions.*

194. A fraction is unaltered in value by multiplying its numerator and denominator by the same number. [Art. 126.]

Hence also a ratio is unaltered in value by multiplying each of its terms by the same number.

Thus the ratios

$$2 : 3,\ \ 6 : 9 \text{ and } 2m : 3m$$

are all equal to one another.

Again, the ratios $4 : 5$, $7 : 9$ and $11 : 15$ are equal respectively to $36 : 45$, $35 : 45$ and $33 : 45$.

Hence the ratios $4 : 5$, $7 : 9$ and $11 : 15$ are in descending order of magnitude.

195. A ratio is altered in value when the same quantity is *added* to each of its terms.

For example, by adding 1, 10 and 100 to each of the terms of the ratio $4 : 5$, we obtain respectively the ratios

$$5 : 6,\ \ 14 : 15 \text{ and } 104 : 105;$$

and these new ratios are different from the given ratio and from each other.

Since

$$\tfrac{4}{5}=1-\tfrac{1}{5},\ \tfrac{5}{6}=1-\tfrac{1}{6},\ \tfrac{14}{15}=1-\tfrac{1}{15} \text{ and } \tfrac{104}{105}=1-\tfrac{1}{105},$$

we see that by adding the same quantity to each of the terms of the ratios $4 : 5$, a new ratio is obtained which becomes more nearly equal to unity as the quantity added becomes greater.

This is a particular case of the following general proposition.

196. *Any ratio is made more nearly equal to unity by adding the same quantity to each of its terms.*

By adding x to each term of the ratio $a : b$, the ratio $a+x : b+x$ is obtained. We have to shew that $\frac{a+x}{b+x}$ is more nearly equal to 1 than is $\frac{a}{b}$.

Now $$\frac{a}{b} - 1 = \frac{a-b}{b}.$$

1/2 3/2 5/4 1 4 2/3

also $$\frac{a+x}{b+x} - 1 = \frac{a-b}{b+x},$$

and it is clear that the *absolute value* of $\frac{a-b}{b+x}$ is less than that of $\frac{a-b}{b}$, for the numerators are the same and the denominator of the first is greater than that of the second: this proves the proposition.

Now when a is greater than b, $\frac{a}{b}$ is greater than 1, and so also is $\frac{a+x}{b+x}$; hence as $\frac{a+x}{b+x}$ is more nearly equal to unity than $\frac{a}{b}$ is, it follows that $\frac{a+x}{b+x}$ is less than $\frac{a}{b}$.

Thus a ratio which is greater than unity is diminished by adding the same quantity to each of its terms.

If however a is less than b, $\frac{a}{b}$ is less than 1, and so also is $\frac{a+x}{b+x}$; but $\frac{a+x}{b+x}$ is more nearly equal to unity than $\frac{a}{b}$ is, and therefore $\frac{a+x}{b+x}$ is greater than $\frac{a}{b}$.

Thus a ratio which is less than unity is increased by adding the same quantity to each of its terms.

Also, when x is very great, the fraction $\frac{a-b}{b+x}$ is very small; and we can make $\frac{a-b}{b+x}$, which is the difference between $\frac{a+x}{b+x}$ and 1, *as small as we please* by taking x sufficiently great. This is expressed by saying that the *limiting value* of $\frac{a+x}{b+x}$ *when* x *is very great*, is unity.

197. The following definitions are sometimes required:

The ratio of the product of the first terms of any number of ratios to the product of their second terms is called the ratio **compounded** of the given ratios.

Thus $ac : bd$ is called the ratio compounded of the ratios

$$a : b \text{ and } c : d.$$

The ratio $a^2 : b^2$ is called the **duplicate ratio** of $a : b$.

The ratio $a^3 : b^3$ is called the **triplicate ratio** of $a : b$.

The ratio $\sqrt{a} : \sqrt{b}$ is called the **sub-duplicate** ratio of $a : b$.

198. **Incommensurable numbers.** The ratio of two quantities cannot always be expressed by the ratio of two whole numbers; for example, the ratio of a diagonal to a side of a square cannot be so expressed, for this ratio is $\sqrt{2}$, and we cannot find any fraction which is *exactly* equal to $\sqrt{2}$.

When the ratio of two quantities cannot be exactly expressed by the ratio of two whole numbers, they are said to be **incommensurable.**

Although the ratio of two incommensurable numbers cannot be found *exactly*, the ratio can be found to any degree of approximation which may be desired; and the different theorems which are proved with respect to ratios of commensurable numbers can be proved to be true also for the ratios of incommensurable numbers.

EXAMPLES. LX.

1. Arrange the ratios $5:6$, $7:8$, $41:48$, and $31:36$ in descending order of magnitude.

2. For what value of x will the ratio $3+x:4+x$ be equal to $5:6$?

3. For what value of x will the ratio $15+x:17+x$ be equal to $\frac{1}{2}$?

4. What must be added to each of the terms of $3:4$ to make the ratio equal to $25:32$?

5. Find two numbers in the ratio of 5 to 6, and whose sum is 121.

6. Two numbers are in the ratio of 3 to 8, and the sum of their squares is 3577: find them.

7. What is the ratio of x to y, if $\dfrac{4x+5y}{3x-y}=2$?

8. If $4x^2+y^2=4xy$, find the ratio of x to y.

9. Find $x:y$, having given $x^2+6y^2=5xy$.

10. A certain ratio will be equal to $2:3$ if 2 be added to each of its terms, and it will be equal to $1:2$ if 1 be subtracted from each of its terms: find the ratio.

11. Find two numbers such that their sum, their difference, and the sum of their squares are as $7:1:75$.

12. What is the least integer which must be added to the terms of the ratio $9:23$ so as to make it greater than the ratio $7:11$?

13. Write down the ratio compounded of the ratios $2:3$ and $15:16$; also the ratio compounded of the ratios $5:6$ and $18:25$.

14. Write down two quantities which are in the duplicate ratio of $2x:3y$.

15. Find x in order that $x+1:x+4$ may be the duplicate ratio of $3:5$.

16. The ages of two persons are as $3:4$ and thirty years ago they were as $1:3$, what are their present ages?

17. Shew that, if from each term of a ratio the inverse of the other be taken, the ratio of the differences will be equal to the original ratio.

18. Shew that, if a and x be positive and $a>x$, then $a^2-x^2:a^2+x^2$ will be greater than $a-x:a+x$.

PROPORTION.

199. **Definition.** Four quantities are said to be **proportional** when the ratio of the first to the second is equal to the ratio of the third to the fourth.

Thus a, b, c, d are proportional, if

$$a : b = c : d.$$

This is sometimes expressed by the notation

$$a : b :: c : d,$$

which is read "a is to b as c is to d."

The first and fourth, of four quantities in proportion, are sometimes called the **extremes**, and the second and third of the quantities are called the **means.**

200. If the four quantities a, b, c, d are proportional, we have by definition

$$\frac{a}{b} = \frac{c}{d}.$$

Multiply each of these equals by bd; then

$$ad = bc.$$

Thus *the product of the extremes is equal to the product of the means.*

Conversely, *if $ad = bc$, then a, b, c, d will be proportional.*

For, if $$ad = bc,$$

then $$\frac{ad}{bd} = \frac{bc}{bd}; \quad \therefore \frac{a}{b} = \frac{c}{d},$$

that is $$a : b = c : d.$$

Thus, if $a : b = c : d$, then $ad = bc$;

and conversely if

$$ad = bc, \text{ then } a : b = c : d.$$

201. It follows from the last article that the four relations

$$a : b = c : d,$$
$$a : c = b : d,$$
$$b : a = d : c,$$

and — $$b : d = a : c,$$

are all true, provided that $ad = bc$.

Hence the above four proportions are all true, when any one of them is true.

202. If $$a : b = c : d,$$

then will $$a + b : a - b = c + d : c - d.$$

For $$a + b : a - b = c + d : c - d,$$

if $$(a + b)(c - d) = (a - b)(c + d),$$

that is, *if* $$ac + bc - ad - bd = ac - bc + ad - bd,$$

or, *if* $$bc = ad.$$

But this condition is satisfied, since $a : b = c : d$.

The above proposition has already been proved in Art. 137, Ex. 1.

203. **Definitions.** Quantities are said to be in **continued proportion** when the ratios of the first to the second, of the second to the third, of the third to the fourth, &c. are all equal.

Thus a, b, c, d &c. are in continued proportion if

$$a : b = b : c = c : d, \text{ \&c.},$$

that is, if $$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \text{\&c.}$$

If $a : b = b : c$, then b is called the **mean proportional** between a and c; also c is called the **third proportional** to a and b.

204. If a, b, c be in continued proportion, we have

$$\frac{a}{b}=\frac{b}{c};$$

$$\therefore \quad b^2=ac, \text{ or } b=\sqrt{ac}.$$

Thus *the mean proportional between two given quantities is the square root of their product.*

Also $$\frac{a}{b}\times\frac{a}{b}=\frac{b}{c}\times\frac{a}{b},$$

that is $$\frac{a^2}{b^2}=\frac{a}{c}.$$

Hence $$a:c=a^2:b^2.$$

Thus, *if three quantities are in continued proportion the ratio of the first to the third is the duplicate ratio of the first to the second.*

205. It is often very convenient to represent a ratio by a single letter, as in Art. 137. The following are additional examples:

Ex. 1. If $$a:b=c:d,$$

then $$a^2+ab:c^2+cd=b^2-2ab:d^2-2cd.$$

Let $$\frac{a}{b}=x; \text{ then also } \frac{c}{d}=x.$$

Hence $$a=bx, \text{ and } c=dx.$$

$$\therefore \quad \frac{a^2+ab}{c^2+cd}=\frac{b^2x^2+b^2x}{d^2x^2+d^2x}=\frac{b^2(x^2+x)}{d^2(x^2+x)}=\frac{b^2}{d^2}.$$

Also $$\frac{b^2-2ab}{d^2-2cd}=\frac{b^2-2b^2x}{d^2-2d^2x}=\frac{b^2(1-2x)}{d^2(1-2x)}=\frac{b^2}{d^2}.$$

Hence $$\frac{a^2+ab}{c^2+cd}=\frac{b^2-2ab}{d^2-2cd},$$

that is $$a^2+ab:c^2+cd=b^2-2ab:d^2-2cd.$$

Ex. 2. If $a:b=c:d=e:f$, shew that

$$a^3+c^3+e^3:b^3+d^3+f^3=ace:bdf.$$

Let $\frac{a}{b}=x$; then $\frac{c}{d}=x$ and $\frac{e}{f}=x$.

Hence $a=bx$, $c=dx$ and $e=fx$;

$$\therefore \quad \frac{a^3+c^3+e^3}{b^3+d^3+f^3}=\frac{b^3x^3+d^3x^3+f^3x^3}{b^3+d^3+f^3}=x^3.$$

And
$$\frac{ace}{bdf}=\frac{bx\,.\,dx\,.\,fx}{bdf}=x^3.$$

Hence
$$\frac{a^3+c^3+e^3}{b^3+d^3+f^3}=\frac{ace}{bdf},$$

that is
$$a^3+c^3+e^3 : b^3+d^3+f^3=ace : bdf.$$

Ex. 3. Shew that, if

$$x : 2b+2c-a=y : 2c+2a-b=z : 2a+2b-c,$$

then will

$$a : 2y+2z-x=b : 2z+2x-y=c : 2x+2y-z.$$

We have
$$\frac{x}{2b+2c-a}=\frac{y}{2c+2a-b}=\frac{z}{2a+2b-c}.$$

Put λ for each of these equal fractions; then

$$x=\lambda\,(2b+2c-a), \quad y=\lambda\,(2c+2a-b), \quad z=\lambda\,(2a+2b-c).$$

Hence $2y+2z-x=9a\lambda$, and similarly

$$2z+2x-y=9b\lambda \text{ and } 2x+2y-z=9c\lambda.$$

Whence
$$\frac{a}{2y+2z-x}=\frac{b}{2z+2x-y}=\frac{c}{2x+2y-z}.$$

206. The definition of proportion given in Euclid is as follows: Four quantities are proportionals, when if any equimultiples be taken of the first and the third, and also any equimultiples of the second and the fourth, the multiple of the third is always greater than, equal to, or less than the multiple of the fourth, according as the multiple of the first is greater than, equal to, or less than the multiple of the second.

If the four quantities a, b, c, d satisfy the algebraical test of proportionality, we have

$$\frac{a}{b}=\frac{c}{d};$$

therefore, for all values of m and n,

$$\frac{ma}{nb} = \frac{mc}{nd}.$$

Hence $\qquad mc \gtreqless nd$, according as $ma \gtreqless nb$.

Thus a, b, c, d satisfy also Euclid's test of proportionality.

Next, suppose that a, b, c, d satisfy Euclid's definition of proportion.

If a and b are commensurable, so that $a : b = m : n$, where m and n are whole numbers; then

$$\frac{a}{b} = \frac{m}{n};$$

$$\therefore \quad na = mb.$$

But, by definition,

$$nc \gtreqless md, \text{ according as } na \gtreqless mb.$$

Hence, as $na = mb$, we must have

$$nc = md;$$

$$\therefore \quad \frac{c}{d} = \frac{m}{n} = \frac{a}{b}.$$

Thus a, b, c, d satisfy the algebraical definition.

If a and b are *incommensurable* we cannot find two *whole numbers* m and n such that $a : b = m : n$. But, if we take any multiple na of a, this must lie between two consecutive multiples, say mb and $(m+1)\,b$, of b, so that

$$na > mb \text{ and } na < (m+1)\,b.$$

Hence, by the definition,

$$nc > md \text{ and } nc < (m+1)\,d;$$

$$\therefore \quad \frac{c}{d} > \frac{m}{n} \text{ and } \frac{c}{d} < \frac{m+1}{n}.$$

Hence both $\frac{a}{b}$ and $\frac{c}{d}$ lie between

$$\frac{m}{n} \text{ and } \frac{m+1}{n}.$$

Thus the difference between $\frac{a}{b}$ and $\frac{c}{d}$ is less than $\frac{1}{n}$; and as this is the case *however great n may be*, $\frac{a}{b}$ must be equal to $\frac{c}{d}$.

EXAMPLES. LXI.

1. Shew that, if $a : b :: c : d$, then

(i) $ac : bd :: c^2 : d^2$.

(ii) $ab : cd :: a^2 : c^2$.

(iii) $a^2 : c^2 :: a^2 - b^2 : c^2 - d^2$.

2. Shew that, if $a : b = c : d$, then

$$2a + 3c : 3a + 2c = 2b + 3d : 3b + 2d.$$

3. If $a : b :: a + c : b + d$, then

$$c : d :: c + a : d + b.$$

4. If $a : b = c : d$, then

$$la + mb : pa + qb = lc + md : pc + qd.$$

5. Shew that, if $3a - 5b : 3c - 5d = 5a + 3b : 5c + 3d$, then

$$a : b = c : d.$$

6. Find a mean proportional to a^3b and ab^3.

7. Find a mean proportional to $(a + b)^2$ and $(a - b)^2$.

8. Find a third proportional to a and a^2; also to

$$(a - b)^2 \text{ and } a^2 - b^2.$$

9. If $a : b :: c : d$, then will $ab + cd$ be a mean proportional between $a^2 + c^2$ and $b^2 + d^2$. 4

10. Shew that, if $a : b :: c : d$, then

(i) $a : a + c :: a + b : a + b + c + d$.

(ii) $a^2 + ab + b^2 : a^2 - ab + b^2 :: c^2 + cd + d^2 : c^2 - cd + d^2$.

(iii) $a+b : c+d :: \sqrt{a^2+b^2} : \sqrt{c^2+d^2}$.

(iv) $\sqrt{a^2+b^2} : \sqrt{c^2+d^2} :: \sqrt[3]{a^3+b^3} : \sqrt[3]{c^3+d^3}$.

(v) $a^2c+ac^2 : b^2d+bd^2 :: (a+c)^3 : (b+d)^3$.

(vi) $\sqrt[n]{a^n+b^n} : \sqrt[n]{c^n+d^n} :: \sqrt[r]{a^r-b^r} : \sqrt[r]{c^r-d^r}$.

11. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, then will $\frac{x+y}{a+b}=\frac{y+z}{b+c}=\frac{z+x}{c+a}$, and also

$$(a^2+b^2+c^2)(x^2+y^2+z^2)=(ax+by+cz)^2.$$

12. Shew that, if

$$\frac{bz-cy}{a}=\frac{cx-az}{b}=\frac{ay-bx}{c}.$$

then will
$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}.$$

13. Shew that, if $\left(\frac{a}{b}\right)^2+\left(\frac{c}{d}\right)^2=2\frac{ac}{bd}$,

then will
$$a : b=c : d.$$

7 **14.** Shew that, if

$$lx(ny-mz),\ my(lz-nx) \text{ and } nz(mx-ly)$$

be equal and not zero, then will

$$mn+nl+lm=0 \text{ and } yz+zx+xy=0.$$

7 **15.** Shew that, if

$$\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c},$$

then will
$$(b-c)x+(c-a)y+(a-b)z=0.$$

16. Shew that, if $a : b=c : d=e : f$, then

$$a^3+a^2c+ace : b^3+b^2d+bdf :: ace+ac^2+c^3 : bdf+bd^2+d^3.$$

17. If $a : x :: b : y :: c : z$, prove that

$$a^4+a^2b^2+b^4 : x^4+x^2y^2+y^4 :: b^4+b^2c^2+c^4 : y^4+y^2z^2+z^4.$$

18. Shew that, if $a : b=pa-qc : pb-qd$, then will

$$c : d=pa+qc : pb+qd.$$

19. Shew that, if $a : b=c : d$, then

$$\frac{a^3}{b}+\frac{b^3}{a} : \frac{c^3}{d}+\frac{d^3}{c}=ab : cd.$$

20. If $4a-b : 4a+b :: 1 : 2$, find the value of

$$7a+3b : 7a-3b.$$

21. If $$xy+3 : xz+1 = y^2+3 : yz+1,$$
shew that $$x=y \text{ or } y=3z.$$

22. Shew that, if $a+b-c : c+d+a = a-c : 2d$,
then $$b : a-c = a+c-d : 2d.$$

23. Shew that, if $x-z : y-z = x^2 : y^2$,
then $$x+z : y+z = x^2+2xy : y^2+2xy.$$

24. Shew that, if $a(y-z)+b(z-x)+c(x-y)=0$,
then $$y-z : b-c = z-x : c-a = x-y : a-b.$$

25. Shew that, if $a : b :: c : d$,
then will
$$a^2+b^2+c^2+d^2 : (a+b)^2+(c+d)^2 :: (a+c)^2+(b+d)^2 : (a+b+c+d)^2.$$

VARIATION.

207. When any substance is sold at a fixed price per pound, the cost of any amount of it is so related to its weight that when the weight is doubled the cost is also doubled, when the weight is halved the cost is also halved, and so on, the ratio of any two values of the cost being equal to the ratio of the corresponding weights. When two quantities are related in this way they are said to *vary as* one another.

Definition. One magnitude is said to **vary as** another when the two are so related that the ratio of any two values of the one is equal to the ratio of the corresponding values of the other.

Thus, if a_1, a_2, be any two measures of one of the quantities, and b_1, b_2, be the corresponding measures of the other, we have

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}; \text{ and therefore } \frac{a_1}{b_1}=\frac{a_2}{b_2}.$$

Hence *the measures of corresponding values of the two magnitudes are always in the same ratio.*

208. The symbol $\propto$ is used for the words *varies as*: thus $A \propto B$ is read 'A varies as B'.

If $a \propto b$, the ratio $a : b$ is constant; and if we put m for this constant ratio, we have

$$\frac{a}{b} = m; \quad \therefore \quad a = mb.$$

To find the constant m in any case it is only necessary to know one set of corresponding values of a and b.

For example, if $a \propto b$, and a is 10 when b is 2, we have

$$a = mb; \quad \therefore \ 10 = m \times 2; \quad \therefore \ m = 5.$$

Hence $a = 5b$.

209. **Definitions.** One quantity is said to **vary inversely as** another when the first varies as the reciprocal of the second.

Thus a varies inversely as b, if $a \propto \frac{1}{b}$, that is if $a = m \times \frac{1}{b}$, or $ab = m$.

One quantity is said to vary as two others **jointly**, when the first varies as the product of the two others.

Thus a varies as b and c jointly if $a \propto bc$, that is if $a = mbc$.

One quantity is said to *vary directly as a second and inversely as a third*, when the ratio of the first to the product of the second and the reciprocal of the third is constant.

Thus a is said to vary directly as b and inversely as c, if $a : b \times \frac{1}{c}$ is constant, that is if $a = m\frac{b}{c}$, where m is a constant.

In all the different cases of variation defined above, the *constant* will be determined when any one set of corresponding values is given.

For example, if a varies as b and inversely as c, and a is 6 when b is 2 and c is 9, we have

$$a = m\frac{b}{c}, \text{ and therefore } 6 = m\frac{2}{9}.$$

Hence $$m = 27, \text{ and therefore } a = 27\frac{b}{c}.$$

210. **Theorem.** If a depends only on b and c, and if a varies as b when c is constant, and varies as c when b is constant; then when both b and c vary, a will vary as bc.

Let a, b, c; a', b', c and a'', b', c' be three sets of corresponding values.

Then, since c is the same in the first and second

$$\frac{a}{a'} = \frac{b}{b'} \qquad \text{(i)}$$

Also, since b' is the same in the second and third cases, we have

$$\frac{a'}{a''} = \frac{c}{c'} \qquad \text{(ii)}.$$

Hence, from (i) and (ii),

$$\frac{a}{a'} \times \frac{a'}{a''} = \frac{bc}{b'c'},$$

that is $$\frac{a}{a''} = \frac{bc}{b'c'},$$

which proves the theorem.

The following are examples of the above proposition.

The cost $[C]$ of a quantity of meat varies as the price per pound $[P]$ if the weight $[W]$ is constant, and the cost varies as the weight if the price per pound is constant; hence, by the proposition, when both the weight and the price per pound change, the cost varies as the product of the weight and the price.

Thus, if $C \propto P$, when W is constant,

and $C \propto W$ when P is constant;

then $C \propto PW$, when both P and W change.

Again the area $[A]$ of a triangle varies as the base $[B]$ when the height $[H]$ is constant; the area also varies as the height when the base is constant; hence, when both the base and the height change, the area will vary as the base and height jointly.

Thus, if $A \propto B$, when H is constant;

and if $A \propto H$, when B is constant;

then $A \propto BH$, when both B and H change.

Again, the pressure $[P]$ of a gas varies as the density $[D]$ when the absolute temperature $[T]$ is constant; the pressure also varies as the absolute temperature when the density is constant; hence, when both density and temperature change, the pressure will vary as the product of the density and temperature.

Thus $P \propto DT$.

Ex. 1. If $A \propto B$, and if also $A \propto C$; then will $B \propto C$.

For, since $A \propto B$, we have $A = mB$, where m is some constant.

And, since $A \propto C$, we have $A = nC$, where n is some constant.

Hence $B = \frac{n}{m} C$, where $\frac{n}{m}$ is some constant; and therefore $B \propto C$.

Ex. 2. If $C \propto WP$, then will $W \propto \frac{C}{P}$.

For, since $C \propto WP$, we have $C = m \, . \, WP$, where m is some constant.

Hence $W = \frac{1}{m}\frac{C}{P}$, where $\frac{1}{m}$ is some constant; therefore $W \propto \frac{C}{P}$.

Ex. 3. The pressure of a gas varies jointly as its density and its absolute temperature; also when the density is 1 and the temperature 300, the pressure is 15. What is the pressure when the density is 3 and the temperature is 320?

Since $$P \propto TD,$$
we have $$P = mTD,$$
where m is some constant.

Also, by the question,

$$15 = m \times 300 \times 1;$$

$$\therefore \quad m = \tfrac{1}{20}.$$

$$\therefore \quad P = \tfrac{1}{20} TD.$$

Hence when D is 3 and T is 320, we have

$$P = \tfrac{1}{20} \times 320 \times 3$$
$$= 48.$$

EXAMPLES. LXII.

1. A varies as B, and A is 5 when B is 3: what is A when B is 5?

2. W varies inversely as P, and W is 4 when P is 15: what is W when P is 12?

3. If $x \propto y$ and $y \propto z$, then will $xz \propto y^2$.

4. If $x^2 \propto y$ and $z^2 \propto y$, then will $xz \propto y$.

5. If $x \propto \dfrac{1}{y}$ and $y \propto \dfrac{1}{z}$, then will $x \propto z$.

6. A varies as B and C jointly; also $A=4$ when $B=2$ and $C=6$: ind the value of A when $B=2$ and $C=9$.

7. A varies as B and inversely as C; also $A=2$ when $B=3$ and $C=4$: find the value of B when $A=6$ and $C=3$.

8. The area of a circle varies as the square of its radius, and the area of a circle whose radius is 10 feet is 314·159 square feet. What s the area of a circle whose radius is 12 feet?

9. The volume of a sphere varies as the cube of its radius, and he volume of a sphere whose radius is 1 foot is 4·188 cubic feet. What is the volume of a sphere whose radius is 3 feet?

10. The velocity of a falling body varies as the time during which t has fallen from rest, and the velocity at the end of two seconds is 4. What is the velocity at the end of five seconds?

11. The distance through which a heavy body falls from rest varies as the square of the time it falls, and a body falls through 144 feet in three seconds. How far does it fall in two seconds?

12. Given that the area of a circle varies as the square of its radius, shew that a circle of 5 feet radius is equal to the sum of a circle of 3 feet radius and another of 4 feet radius.

13. The volume of a gas varies as the absolute temperature and inversely as the pressure. Also when the pressure is 15 and the temperature 280 the volume is 1 cubic foot; what is the volume when the pressure is 20 and the temperature 300?

14. If a^2-b^2 varies as c^2, and if $c=2$ when $a=5$ and $b=3$: find the equation between a, b and c, and shew that b is a mean proportional between $a-2c$ and $a+2c$.

15. The volume of a right circular cone varies jointly as its height and the square of the radius of its base; and the volume of a cone 7 feet high with a base whose radius is 3 feet is 66 cubic feet. Find the volume of a cone 9 feet high with a base whose radius is 14 feet.

16. The volume of a sphere varies as the cube of its radius: if three spheres of radii 6, 8 and 10 inches respectively be melted and formed into a single sphere, find its radius.

17. The distance of the offing at sea varies as the square root of the height of the eye above the sea level, and the distance is 3 miles when the height is 6 feet: find the distance when the height is 50 yards.

MISCELLANEOUS EXAMPLES. V.

A. 1. If $\frac{2a}{3}=\frac{3b}{4}=\frac{c}{2}=1$, find the numerical value of

$$(a-b)^2+(b-c)^2+(c-a)^2.$$

2. Divide

$$9a^2b^3-12a^4b+3b^5+2a^3b^2+4a^5-11ab^4 \text{ by } 3b^3+4a^3-2ab^2.$$

3. Find the H.C.F. of $1-x-x^3+x^5$ and $1-x^4-x^6-x^7$.

4. Simplify $\dfrac{a^2 - a^2b^{-2} - 1 + b^{-2}}{a + ab^{-1} + 1 + b^{-1}}$.

5. A person bought 15 ducks and 12 geese for £5. 5*s.*; and the rices were such that 2 more ducks could be bought for 18*s.*, than eese for £1. What was the price of a duck?

6. Shew that the difference of the roots of the equation $^2 - px + q = 0$ is equal to the difference of the roots of the equation $^2 - 3px + 2p^2 + q = 0$.

7. What is the least integer which must be added to the terms of he ratio 3 : 4 to make it greater than the ratio 19 : 21?

8. Shew that, if $a+b$, $b+c$, $c+a$ are in continued proportion, hen
$$b+c : c+a :: c-a : a-b.$$

1. Find the value of
$$(xy + \sqrt{1-x^2}\sqrt{1-y^2}) \div (x\sqrt{1-y^2} - y\sqrt{1-x^2}),$$
hen $x = \frac{4}{5}$, $y = \frac{3}{5}$.

2. Multiply $x^2 + (a-1)x + a + 1$ by $(a-1)x - a^2 - a - 1$.

3. Find the factors of (i) $x^3 - 13x^2y + 42xy^2$,
(ii) $(a+2b+3c)^2 - 4(a+b-c)^2$, and (iii) $x^2 - 4xy + 4y^2 - 9$.

4. Simplify $\dfrac{x}{x-2y} + \dfrac{3x}{x+2y} + \dfrac{4xy}{4y^2 - x^2}$.

5. Solve the equations:

(i) $\dfrac{10}{x} - \dfrac{3}{x+2} = \dfrac{10}{x+1}$.

(ii) $\left.\begin{aligned} x + y + \sqrt{(x+y)} &= 12 \\ x^2 - y^2 &= 21 \end{aligned}\right\}$.

6. If $x + \dfrac{1}{y} = 1$ and $y + \dfrac{1}{z} = 1$, prove that $z + \dfrac{1}{x} = 1$, and that $yz + 1 = 0$.

7. Find the square root of
$$9x^6 - 12x^4y^2 + 30x^3y^3 + 4x^2y^4 - 20xy^5 + 25y^6.$$

8. Find two numbers such that their sum, their difference, and he sum of their squares are as 3 : 1 : 15.

9. If $a : b :: c : d$, shew that
$$la + mb : lc + md :: \sqrt{a^3 + b^2d} : \sqrt{ac^2 + d^3}.$$

C. 1. Subtract $a-2(b-c)$ from $3\{b+2(a-c)-5(a-b)\}$.

2. Multiply $a+b+\frac{b^2}{a}+\frac{a^2}{b}$ by $a-b+\frac{b^2}{a}-\frac{a^2}{b}$.

3. Multiply $\frac{a^2+b^2}{9(a^2-b^2)}$ by $27(a-b)$, and divide

$$\frac{9a^2-16b^2}{a+b} \text{ by } \frac{3a-4b}{a^2-b^2}.$$

4. If $a=y+z$, $b=z+x$, $c=x+y$, then

$$a^2+b^2+c^2-bc-ca-ab=x^2+y^2+z^2-yz-zx-xy.$$

5. Find the square root of

$$a^{\frac{4}{3}}+4ay^{\frac{1}{5}}+10a^{\frac{2}{3}}y^{\frac{2}{5}}+12a^{\frac{1}{3}}y^{\frac{3}{5}}+9y^{\frac{4}{5}}.$$

X 6. If $x=2+\sqrt{2}$, prove that $(x-1)(x-2)=x$.

7. A and B start simultaneously from two towns to meet each other: A travels two miles an hour slower than B and they meet in seven hours: if B had travelled one mile an hour faster than he did and A at only half his previous pace, they would have met in nine hours. Find the distance between the towns.

8. If $a(y+z)=b(z+x)=c(x+y)$; then

$$\frac{y-z}{a(b-c)}=\frac{z-x}{b(c-a)}=\frac{x-y}{c(a-b)}.$$

D. 1. Shew that $\left(\frac{a^3-b^3}{a-b}\right)^2-\left(\frac{a^3+b^3}{a+b}\right)^2=4ab(a^2+b^2)$.

2. Divide

$x^4-2bx^3-(a^2-b^2)x^2+2a^2bx-a^2b^2$ by $x^2-(a+b)x+ab$.

3. Find the L.C.M. of $x^2-7x+12$, $3x^2-6x-9$ and $2x^3-6x^2-8x$.

4. Simplify $\dfrac{\frac{y}{x}-\frac{y}{y+z}}{\frac{1}{xz}+\frac{1}{zy+z^2}}\left\{1+\frac{y^2+z^2-x^2}{2yz}\right\}$.

5. Solve the equations:

(i) $\frac{5x+7}{2}-\frac{2x-7}{3}=3x+14$.

(ii) $\left.\begin{array}{l} y^2-xy=15 \\ x^2+xy=14 \end{array}\right\}$.

6. The fore-wheel of a carriage makes 64 revolutions more than the hind-wheel in travelling one mile, and the sum of the revolutions made by each in a journey of 10 miles is 7040: find the circumference of each wheel.

7. Divide $2x^{\frac{3}{2}} - 17xy^{\frac{1}{2}} + 17x^{\frac{1}{2}}y - 15y^{\frac{3}{2}}$ by $2x^{\frac{1}{2}} - 15y^{\frac{1}{2}}$.

8. Prove that if the ratios $a : \alpha$, $b : \beta$ and $c : \gamma$ are all equal, then each is equal to the ratio $a+b+c : \alpha+\beta+\gamma$.

E. 1. Simplify the expression

$$2\{x - 2a - \tfrac{2}{3}(x-a) + a\} - \tfrac{1}{3}\{(x-a) - (a-b) + y - b\}.$$

2. Divide $2a^2x^2 - 2(3b-4c)(b-c)y^2 + abxy$ by $ax + 2(b-c)y$.

3. Find the factors of the following expressions:

(i) $4a^2b^4 - 16a^4b^2$, (ii) $x^2 - 26x - 87$, and (iii) $3x^2 - xy - 10y^2$.

4. Shew that

$$\frac{1}{(y-z)^2} + \frac{1}{(z-x)^2} + \frac{1}{(x-y)^2} = \left(\frac{1}{y-z} + \frac{1}{z-x} + \frac{1}{x-y}\right)^2.$$

5. A man bought a number of articles for £21. If he had got six more for the same money each would have cost 1*s.* 3*d.* less than it did; how many did he buy?

6. Multiply $\sqrt{(x-a)} + \sqrt{a} - \sqrt{x}$ by $\sqrt{(x-a)} - \sqrt{a} + \sqrt{x}$.

7. Shew that the mean proportional between $x^2 - \dfrac{1}{y^2}$ and $y^2 - \dfrac{1}{x^2}$ is $xy - \dfrac{1}{xy}$.

8. If $a : b :: c : d$, shew that

$$a^2 + b^2 : c^2 + d^2 :: (a+b)^2 : (c+d)^2.$$

F. 1. Multiply $a - b + c - d$ by $a + b - c + d$.

2. Divide $a^3 + b^3 + c^3 + 3(b+c)(c+a)(a+b)$ by $a + b + c$.

3. Find the H.C.F. of $x^3 - 19x + 30$ and $5x^3 - 19x^2 + 36$. For what values of x will both expressions vanish?

4. Shew that $\left(\dfrac{a+b}{a-b}\right)^2 - \left(\dfrac{a-b}{a+b}\right)^2 = \dfrac{8ab(a^2+b^2)}{(a^2-b^2)^2}$.

5. Solve the equations :

(i) $$\frac{6x-7}{9x+6}-\frac{5(x-1)}{12x+8}=\frac{1}{2}.$$

(ii) $$\left.\begin{aligned} 4x+6y&=3 \\ 4x^2+9xy+9y^2&=11 \end{aligned}\right\}.$$

6. Simplify $$\frac{2\sqrt{(\frac{1}{3})}+3\sqrt{(\frac{1}{2})}}{\frac{1}{2}\sqrt{(\frac{1}{3})}-\frac{1}{3}\sqrt{(\frac{1}{2})}},$$

and $$x^{\frac{1}{2}}y^{\frac{1}{3}}\times\left(\frac{x^{-\frac{2}{3}}}{y}\right)\div\{x^{-\frac{1}{6}}y^{-\frac{1}{4}}\}.$$

7. Find the square root of

$$x^{12}-6x^{10}+13x^8-14x^6+10x^4-4x^2+1.$$

8. If $a:b::c:d$; shew that $a^3c^2 : b^3d^2 :: a^5+c^5 : b^5+d^5$.

CHAPTER XXII.

ARITHMETICAL PROGRESSION.

211. **Def.** A series of quantities is said to be in **Arithmetical Progression** when the difference between any term and the *preceding* one is the same throughout the series.

Thus a, b, c, d, &c. are in Arithmetical Progression (A.P.) if

$$b - a = c - b = d - c, \text{ \&c.}$$

The difference between each term of an A.P. and the preceding term is called the **common difference.**

The following are examples of arithmetical progressions:—

$$\begin{array}{lrrrrl} & 1, & 3, & 5, & 7, & \text{\&c.}, \\ & 2, & 6, & 10, & 14, & \text{\&c.}, \\ & 9, & 8, & 7, & 6, & \text{\&c.}, \\ \text{and} & 3, & -1, & -5, & -9, & \text{\&c.} \end{array}$$

In the first series the common difference is 2, in the second series it is 4, in the third it is -1, and in the last it is -4.

212. If the first term of an arithmetical progression be a, and the common difference d; then

the 2nd term will be $a + d$,
... 3rd .... $a + 2d$,
... 4th .... $a + 3d$,

and so on, the coefficient of d being always less by unity than the number giving the position of the term in the series.

Thus the nth term is

$$a+(n-1)\,d.$$

We can therefore write down any term of an A.P. when the first term and the common difference are given.

For example, in the A.P. whose first term is 5, and whose common difference is 3,

the 9th term is $5+(9-1)\,3=29$.
and the 27th term is $5+(27-1)\,3=83$.

213. When *any two* terms of an A.P. are given we can find the first term and the common difference, and therefore any other term of the series.

Suppose, for example, that the 10th term of an A.P. is 25, and the 15th term is 5.

Let the first term be a, and the common difference d.

Then the 10th term will be $a+9d$,
and the 15th term will be $a+14d$.

Hence $a+9d=25$,
and $a+14d=5$.

By subtraction $5d=-20$; $\therefore\ d=-4$.

Then $a=25-9d=25-9(-4)=61$.

Thus the series is 61, 57, 53, &c.

Ex. The 12th term of an A.P. is 15 and the 19th term is 36; find the 30th term.

Let the first term be a and the common difference d.

Then the 12th term will be $a+11d$, and the 19th term will be $a+18d$.

Hence by the question

$$a+11d=15,$$
$$a+18d=36.$$

By subtraction $7d=21$; $\therefore\ d=3$.

Then $a=15-11d=15-33=-18$.

Hence the 30th term $=a+29d=-18+29\times 3=69$.

214. When three quantities are in arithmetical progression, the middle one is called the **Arithmetic Mean** of the other two.

Thus, if a, b, c are in A.P., b is the arithmetic mean of a and c.

By the definition of arithmetical progression, we have

$$b - a = c - b;$$

$$\therefore\ b = \tfrac{1}{2}(a + c).$$

Thus *the arithmetic mean of any two quantities is half their sum.*

215. When any number of quantities are in arithmetical progression all the intermediate terms may be called **arithmetic means** of the two extreme terms.

Between any two given quantities any number of arithmetic means may be inserted.

For example, to insert four arithmetic means between 10 and 25.

We have to find an A.P. with *four* terms between 10 and 25, so that 10 is the first and 25 is the *sixth* term of an A.P.

Let d be the common difference.

Then the sixth term is

$$10 + 5d;$$

$$\therefore\quad 10 + 5d = 25;\ \therefore\ d = 3.$$

Thus the series is

$$10,\quad 13,\quad 16,\quad 19,\quad 22,\quad 25;$$

and the required arithmetic means between 10 and 25 are

$$13,\quad 16,\quad 19,\quad 22.$$

We now consider the most general case, namely to insert n arithmetic means between a and b.

We have to find an A.P. with n terms between a and b, so that a is the first and b is the $(n + 2)$th term of the A.P.

Let d be the common difference.

Then the $(n+2)$th term is $a+(n+1)d$;

$$\therefore \quad a+(n+1)d=b,$$

$$\therefore \qquad (n+1)d=b-a;$$

$$\therefore \qquad d=\frac{b-a}{n+1}.$$

Thus the series is

$$a, \quad a+\frac{b-a}{n+1}, \quad a+2\frac{b-a}{n+1}\dots, \quad a+n\frac{b-a}{n+1}, \quad b;$$

and the arithmetic means required are

$$a+\frac{b-a}{n+1}, \quad a+2\frac{b-a}{n+1}, \quad a+3\frac{b-a}{n+1}, \dots\dots, \quad a+n\frac{b-a}{n+1},$$

that is

$$\frac{na+b}{n+1}, \quad \frac{(n-1)a+2b}{n+1}, \quad \frac{(n-2)a+3b}{n+1}, \dots\dots, \quad \frac{a+nb}{n+1}.$$

EXAMPLES. LXIII.

1. Find the 30th terms of each of the following arithmetical progressions:

(i) 3, 5, 7, &c. (ii) 1, 5, 9, &c. (iii) 12, 9, 6, &c.

(iv) $\frac{1}{9}, \frac{1}{3}, \frac{5}{9}$, &c. (v) $a+b, a, a-b$, &c.

2. Find the last term of each of the following series:

(i) 3, 6, 9, &c. to 24 terms. (ii) 5, 9, 13, &c. to 30 terms.

(iii) 6, 5, 4, &c. to 10 terms. (iv) 14, 46, 78, &c. to 12 terms.

(v) 6, $8\frac{2}{3}$, $11\frac{1}{3}$, &c. to 14 terms. (vi) $\frac{1}{3}, -\frac{1}{2}, -\frac{4}{3}$, &c. to 25 terms.

3. The 10th term of an A.P. is 6 and the 6th term is 10. Find the first term.

4. The 12th term of an A.P. is 15 and the 20th term is 25. Find the common difference.

5. The 7th term of an A.P. is 5 and the 12th term is 30. Find the common difference.

6. The 3rd term of an A.P. is 40 and the 13th term is 25. Find the first term.

7. What is the 10th term of the A.P. whose first term is 7 and whose third term is 13?

8. What is the 12th term of the A.P. whose first term is 20 and whose sixth term is 10?

9. The 3rd term of an A.P. is 10 and the 14th term is 54: find the 20th term.

10. The 7th term of an A.P. is 5, and the 5th term is 7. What is the 12th term?

11. Which term of the series 5, 8, 11, &c. is 65?

12. Which term of the series $\frac{7}{6}$, $\frac{4}{3}$, $\frac{3}{2}$, &c. is 18?

13. Which term of the series 9, 13, 17, &c. is 229?

14. Which term of the A.P. $16a-8b$, $15a-7b$, $14a-6b$, &c. is $8a$?

15. Write down the arithmetic mean of (i) 7 and 13; (ii) 9 and -9, and (iii) $a+b$ and $a-b$.

16. Insert 6 arithmetic means between 8 and 29.

17. Insert 8 arithmetic means between 50 and 80.

18. Insert 7 arithmetic means between 269 and 295.

19. Insert 15 arithmetic means between 67 and 43.

20. Insert 25 arithmetic means between 84 and $40\frac{2}{3}$.

21. Insert 10 arithmetic means between $5a-6b$ and $5b-6a$.

22. Insert 8 arithmetic means between $a-5b$ and $b-5a$.

23. If a, b, c, d are in A.P., shew that $a+d=b+c$.

24. The sum of the first and fourth terms of an A.P. is 19, and the sum of the third and sixth terms is 31. What is the first term?

25. The sum of the second and fifth terms of an A.P. is 32, and the sum of the third and eighth terms is 48. What is the first term?

26. The sum of the third and fourth terms of an A.P. is 187, and the sum of the seventh and eighth terms is 147. What is the second term?

27. The sum of the second and twentieth terms of an A.P. is 2, and the sum of the ninth and fifteenth terms is 8. What is the sum of the sixth and seventh terms?

28. Shew that, if the same quantity be added to every term of an A.P., the sums will be in A.P.

29. Shew that, if every term of an A.P. be multiplied by the same quantity, the products will be in A.P.

30. Shew that, if every alternate term of an A.P. be taken away, the remaining terms will be in A.P.

31. Shew that, if between every two consecutive terms of an A.P. their arithmetic mean be inserted, the whole will form another arithmetical progression.

32. Shew that, if four quantities are in A.P., the product of the first and fourth is always less than the product of the second and third.

216. *To find the sum of any number of terms of an arithmetical progression.*

Let a be the first term and d the common difference. Let n be the number of the terms whose sum is required, and let l be the last term.

Then, since l is the nth term, we have

$$l = a + (n-1)d.$$

Hence, if S be the required sum,

$$S = a + (a+d) + (a+2d) + \dots + (l-2d) + (l-d) + l.$$

Now write the series in the reverse order; then

$$S = l + (l-d) + (l-2d) + \dots + (a+2d) + (a+d) + a.$$

Hence, by addition of corresponding terms, we have

$$\begin{aligned} 2S &= (a+l) + (a+l) + (a+l) + \dots \text{ to } n \text{ terms} \\ &= n(a+l); \end{aligned}$$

$$\therefore S = \frac{n}{2}(a+l) \quad \dots\dots\dots\dots\dots(\text{i}).$$

But $a+l=a+\{a+(n-1)d\}=2a+(n-1)d$;

$$\therefore\ S=\frac{n}{2}\{2a+(n-1)d\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

The formulae (i) and (ii) are both important and should be remembered.

It should be remarked that the formula (ii) gives the value of any one of the four quantities a, d, n and S when the other three are known.

Ex. 1. Find the sum of the first 20 terms of the series

$$5+8+11+\&c.$$

Here $a=5,\quad d=3,\quad n=20$;

$$\therefore\quad S=\frac{n}{2}\{2a+(n-1)d\}$$

$$=\frac{20}{2}\{10+19\times 3\}=670.$$

Ex. 2. Shew that the sum of any number of consecutive odd numbers, beginning with unity, is a square number.

The series of odd numbers is

$$1+3+5+7+\ldots\ldots$$

Here $a=1$ and $d=2$; hence the sum of n terms is given by

$$S=\frac{n}{2}\{2a+(n-1)d\}$$

$$=\frac{n}{2}\{2+(n-1)2\}=\frac{n}{2}\times 2n=n^2.$$

Thus the sum of n consecutive odd numbers beginning with unity is n^2.

Ex. 3. The sum of 20 terms of an arithmetical progression is 410, and the first term is 30. What is the common difference?

We have $S=\frac{n}{2}\{2a+(n-1)d\}$,

where $S=410,\quad a=30$, and $n=20$.

Hence $410=\frac{20}{2}\{60+19d\}$.

From which we find that $d=-1$.

Ex. 4. How many terms of the series $11 + 12 + 13 + \&c.$ must be taken in order that the sum may be 410?

We have $$S = \frac{n}{2}\{2a + (n-1)d\},$$

where $a = 11$, $d = 1$, and $S = 410$.

Hence $$410 = \frac{n}{2}\{22 + (n-1)\};$$

$$\therefore\ n^2 + 21n - 820 = 0,$$

that is $$(n-20)(n+41) = 0;$$

$$\therefore\ n = 20, \text{ or } n = -41.$$

When a, S and d are given, n is to be found by solving a *quadratic* equation, and one of the roots of the equation is generally inapplicable, for any value of n which is not a *positive integer* is without meaning.

In the present case, -41 is to be rejected*: thus the only value of n is 20.

Ex. 5. How many terms of the series $24 + 21 + 18 + \&c.$ must be taken in order that the sum may be 105?

We have $$S = \frac{n}{2}\{2a + (n-1)d\},$$

where $a = 24$, $d = -3$, and $S = 105$.

Hence $$105 = \frac{n}{2}\{48 + (n-1)(-3)\};$$

$$\therefore\ 210 = n\{48 - 3n + 3\};$$

$$\therefore\ n^2 - 17n + 70 = 0.$$

Hence $$n = 7, \text{ or } n = 10.$$

Since *both* values of n are positive integers, they are both answers to the question. Whenever the sum of an A.P. is the same for two values of n the additional terms for the greater value have a zero sum; in the present case these additional terms are 3, 0, -3.

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* The beginner should be cautioned against the error of supposing that -41 might mean 41 terms reckoned *backwards*.

EXAMPLES. LXIV.

Find the sum of the following series:

1. $2+4+6+\dots$ to 20 terms.

2. $15+14\frac{1}{2}+14+\dots$ to 16 terms.

3. $1+2\frac{1}{4}+3\frac{1}{2}\dots$ to 12 terms.

4. $-5-1+3+\dots$ to 20 terms.

5. $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}+\dots$ to 7 terms.

6. $\frac{1}{2}-\frac{2}{3}-\frac{11}{6}-\dots$ to 61 terms.

7. $10+\frac{29}{3}+\frac{28}{3}+\dots$ to 7 terms.

8. $\frac{7}{6}+1+\frac{5}{6}+\dots$ to 15 terms.

9. $3\frac{1}{3}+2\frac{1}{2}+1\frac{2}{3}+\dots$ to n terms.

10. $1\frac{1}{7}+1\frac{17}{21}+2\frac{10}{21}+\dots$ to n terms.

11. $8+7\frac{1}{9}+6\frac{2}{9}+\dots$ to 19 terms.

12. $4\frac{1}{2}+4\frac{1}{5}+3\frac{9}{10}+\dots$ to 31 terms.

13. $5+6{\cdot}2+7{\cdot}4+\dots$ to 21 terms.

14. $\dfrac{1}{\sqrt{2}+1}+\sqrt{2}+\dfrac{1}{\sqrt{2}-1}+\dots$ to 7 terms.

15. $\dfrac{n-1}{n}+\dfrac{n-2}{n}+\dfrac{n-3}{n}+\dots$ to n terms.

16. $n+1+(2n+3)+(3n+5)+\dots$ to n terms.

17. $(a+b)^2+(a^2+b^2)+(a-b)^2+\dots$ to n terms.

18. The 3rd term of an arithmetical progression is 15, and the 20th term is $23\frac{1}{2}$; find the sum of the first 20 terms.

19. The 5th term of an A.P. is 37 and the 13th term is 81; find the sum of the first 24 terms.

20. Find the sum of 20 consecutive odd numbers of which the least is 25.

21. Find the sum of 40 consecutive odd numbers of which the greatest is 99.

22. Insert 29 arithmetic means between 5 and 50, and find their sum.

23. Insert 40 arithmetic means between 10 and 100, and find their sum.

24. There are 27 terms of an A.P. of which 3 is the first and 107 is the last; find the middle term and the sum of all the terms.

25. There are 71 terms of an A.P. of which the first is 7 and the last is 1015; find the middle term and the sum of all the terms.

26. There are 19 terms in A.P. of which the first is 8 and the last -4; find the sum of the series.

27. Find the sum of 20 terms of the series 3, 5, 7, &c. beginning at the seventh.

28. Find the sum of 35 terms of the series 6, 9, 12, &c., beginning at the fifth.

29. The sum of 10 terms of an A.P., whose first term is 2, is 155: what is the common difference?

30. The sum of 25 terms of an A.P., whose first term is 6, is 25; what is the common difference?

31. The sum of 10 terms of an A.P. is 100, and the sixth term is 11; what is the first term?

32. The sum of 28 terms of an A.P. is 133, and the fifth term is 0; what is the common difference?

33. The sum of 10 consecutive terms of the series 3, 8, 13, &c. is 705: which is the first of them?

34. The sum of 25 successive terms of the series 5, 8, 11,... is 1025; which is the first of them?

35. How many terms of the series $\frac{4}{3}$, 1, $\frac{2}{3}$, must be taken in order that the sum may be zero?

36. How many terms of the series 15, 12, 9,... must be taken in order that the sum may be 45?

37. How many terms of the series $1-\frac{1}{a}$, $1-\frac{3}{a}$, $1-\frac{5}{a}$,... must be taken in order that the sum may be $-6a$?

38. How many terms of the series $-8-7-6-\ldots$ will amount to 42?

39. The first term of a certain A.P. is 1, the last is 99, and the sum of all the terms is 450: how many terms are there?

40. The last term of an A.P. of 20 terms is 62 and the sum is 670: what is the common difference?

41. The ninth term of an A.P. is 136 and the sum of the first 19 terms is 2527. Find the sum of the first 40 terms.

42. Find the sum of 15 terms of an arithmetical progression of which the eighth is 6.

43. Find the sum of 35 terms of an arithmetical progression of which the eighteenth term is 15.

35 16

44. Find the sum of $2n+1$ terms of an A.P. whose $(n+1)$th term is 100.

45. Shew that, if any odd number of quantities are in A.P., the first, the middle, and the last are also in A.P.

46. Shew that, if unity be added to the sum of any number of terms of the series 8, 16, 24, &c., the result will be the square of an odd number.

47. Find the sum of all the odd numbers between 100 and 200.

48. Find the sum of all the even numbers which are between 101 and 999.

49. Find the sum of all the numbers between 100 and 500 which are divisible by 3.

50. A man saved each year £10 more than he did in the preceding year, and he saved £20 the first year: in how many years will his savings amount to £1700?

51. The sum of 10 terms of an arithmetical series is 145, and the sum of its fourth and ninth terms is five times the third term; determine the series.

52. In an A.P. consisting of $2n+1$ terms, shew that the sum of the odd terms is to the sum of the even terms as $n+1 : n$.

53. In an A.P. consisting of 21 terms, the sum of the three last terms is 117, and the sum of the three middle terms is 90; find the series.

54. The sum of n terms of the series 87, 85, 83, &c. is equal to the sum of n terms of the series 3, 5, 7, &c. Find n.

55. The sum of n terms of the A.P. 43, 45, 47, &c. is equal to the sum of $2n$ terms of the A.P. 45, 43, 41, &c. Find n.

56. Divide 80 into four parts which are in A.P., and which are such that the product of the first and fourth is two-thirds of the product of the second and third.

57. Find four numbers in A.P. such that the sum of their squares shall be 120, and that the product of the first and last shall be less than the product of the other two by 8.

58. If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ be in A.P., prove that $(s-a)^2$, $(s-b)^2$, $(s-c)^2$ are also in A.P., where $2s=a+b+c$.

59. Shew that, if a, b, c be in A.P., then will $a^2(b+c)$, $b^2(c+a)$, $c^2(a+b)$ also be in A.P.

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CHAPTER XXIII.

GEOMETRICAL PROGRESSION.

217. **Def.** A series of quantities is said to be in **Geometrical Progression** when the ratio of any term to the preceding one is the same throughout the series.

Thus a, b, c, d, &c. are in Geometrical Progression (G.P.) if $\frac{b}{a}=\frac{c}{b}=\frac{d}{c}$, &c.

The ratio of each term of a geometrical progression to the preceding term is called the **common ratio.**

The following are examples of geometrical progressions:

$$1,\quad 3,\quad 9,\quad 27,\ \&c.$$
$$4,\quad 2,\quad 1,\quad \tfrac{1}{2},\ \&c.$$
and
$$\tfrac{2}{3},\quad -1,\quad \tfrac{3}{2},\quad -\tfrac{9}{4},\ \&c.$$

In the first series the common ratio is 3, in the second series it is $\frac{1}{2}$, and in the last it is $-\frac{3}{2}$.

218. If the first term of a G.P. be a, and the common ratio r; then

the 2nd term will be ar,
the 3rd .... ar^2,
the 4th .... ar^3,

and so on, the index of r being always less by unity than the number giving the position of the term in the series.

Thus the nth term is ar^{n-1}.

We can therefore write down any term of a G.P. when the first term and the common ratio are given.

For example in the G.P. whose first term is 5, and whose common ratio is 3,

the 4th term is $5 \times 3^{4-1} = 5 \times 3^3$,

and the 10th term is $5 \times 3^{10-1} = 5 \times 3^9$.

219. When any *two* terms of a G.P. are given, we can find the first term and the common ratio: the series will thus be completely determined.

Suppose, for example, that the fifth term of a G.P. is $\frac{8}{9}$, and the 7th term $\frac{32}{81}$.

Let the first term be a, and the common ratio r.

Then the 5th and the 7th terms are ar^4 and ar^6 respectively.

Hence $ar^4 = \frac{8}{9}$,

and $ar^6 = \frac{32}{81}$;

$\therefore$ by division, $r^2 = \frac{4}{9}$;

$\therefore r = \pm\frac{2}{3}$.

Then, since $a \times \frac{16}{81} = \frac{8}{9}$, $a = \frac{9}{2}$.

Thus the series is $\frac{9}{2}$, ± 3, 2, $\pm\frac{4}{3}$, &c.

Ex. The 4th term of a G.P. is 189 and the 6th term is 1701, find the 8th term.

Let the first term be a, and the common ratio r; then the 4th term will be ar^3 and the 6th term will be ar^5.

Hence, by the question,

$$ar^3 = 189 \text{ and } ar^5 = 1701.$$

By division $r^2 = 9$; $\therefore r = \pm 3$.

Then $a = 189 \div (\pm 3)^3 = \pm 7$.

Hence the 8th term $= ar^7 = \pm 7\,(\pm 3)^7 = 15309$.

220. When three quantities are in geometrical progression, the middle one is called the **geometric mean** of the other two.

Thus, if a, b, c are in G.P., b is the geometric mean of a and c. By the definition of geometric progression, we have

$$\frac{b}{a} = \frac{c}{b};$$

$$\therefore\ b^2 = ac;$$

$$\therefore\ b = \pm\sqrt{ac}.$$

Thus *the geometric mean of any two quantities is the square root of their product.*

It should be noticed that quantities which are in continued proportion [Art. 203], are in geometrical progression.

221. When any number of quantities are in geometrical progression all the intermediate terms may be called **geometric means** of the two extreme terms.

Between two given quantities any number of geometric means may be inserted.

For example, to insert three geometric means between 5 and 80.

We have to find a G.P. with *three* terms between 5 and 80, so that 5 is the first term and 80 is the fifth term.

Let r be the common ratio; then the fifth term is ar^4.

Hence $5r^4 = 80;\ \therefore\ r^4 = 16;\ \therefore\ r = \pm 2.$

Thus the series is 5, ± 10, 20, ± 40, 80.

The required means are therefore ± 10, 20, ± 40.

We now consider the most general case, namely to insert n geometric means between a and b.

We have to find a G.P. with n terms between a and b, so that a is the first and b is the $(n+2)$th term of the G.P.

Let r be the common ratio. Then the $(n+2)$th term is ar^{n+1};

$$\therefore\ ar^{n+1} = b;$$

$$\therefore\ r^{n+1} = \frac{b}{a};$$

$$\therefore\ r = \sqrt[n+1]{\frac{b}{a}}.$$

Hence the required means are ar, ar^2, ar^3.........ar^n, where $r = \sqrt[n+1]{\dfrac{b}{a}}$.

EXAMPLES. LXV.

1. Find the 6th terms of each of the following geometrical progressions:

(i) 9, 3, 1, &c. (ii) 2, -3, $\frac{9}{2}$, &c. (iii) a^2, ab, b^2, &c.

2. What is the fifth term of the G.P. whose first term is 3 and whose third term is 4?

3. The 3rd term of a G.P. is 1, and the 6th term is $-\frac{1}{8}$: what is the 10th term?

4. The 4th term of a G.P. is ·016 and the 7th term is ·000128: what is the first term?

5. The 6th term of a G.P. is 156 and the 8th term is 7644: what is the 7th term?

6. The 3rd term of a G.P. is 2·25 and the 7th term is 11·390625. What is the 5th term?

7. The 3rd term of a G.P. is 4 and the 6th term is $-\frac{1}{2}$. What is the 10th term?

8. The 4th term of a G.P. is $\frac{1}{18}$ and the 7th term is $-\frac{1}{486}$. What is the 6th term?

9. Find the geometric mean (i) of 4 and 9, (ii) of 7 and 252, and (iii) of a^3b and ab^3.

10. Insert two geometric means between 1 and -8; also insert three geometric means between 12 and $\frac{3}{4}$, and four between $\frac{2}{3}$ and $\frac{81}{16}$.

11. Insert five geometric means between 3 and ·000192.

12. Insert four geometric means between a^3b^{-6} and $a^{-2}b^4$.

13. The sum of the third and fourth terms of a G.P. is 40, and the sum of the sixth and seventh terms is 2560. What is the first term?

14. The sum of the first and second terms of a G.P. is 72 and the sum of the third and fourth terms is 8. What is the first term?

15. If a, b, c, d are in G.P., shew that $ad=bc$.

16. If all the terms of a G.P. be multiplied by the same quantity, the products will be in G.P.

17. Shew that the reciprocals of the terms of a geometrical progression are in G.P.

18. If every alternate term of a G.P. be taken away, the remaining terms will be in G.P.

19. If between every two consecutive terms of a G.P. their geometric mean be inserted, the whole will form another geometrical progression.

20. Shew that the product of any two terms of a geometrical progression, which are respectively equally distant from the first and the last terms, is equal to the product of the first and last terms.

222. To find the sum of any number of terms of a geometrical progression.

Let a be the first term, and r the common ratio. Let n be the number of the terms whose sum is required, and let l be the last term.

Then, since l is the nth term, we have

$$l = ar^{n-1}.$$

Hence, if S be the required sum,

$$S = a + ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1}.$$

Multiply by r; then

$$Sr = ar + ar^2 + ar^3 + \ldots \qquad + ar^{n-1} + ar^n.$$

Hence, by subtraction,

$$S - Sr = a - ar^n;$$

that is

$$S(1-r) = a(1-r^n);$$

$$\therefore \quad S = \frac{a(1-r^n)}{1-r}.$$

The above formula may be written in another form; for, by substituting the value $l = ar^{n-1}$, we have

$$S = \frac{a - rl}{1-r}.$$

Ex. 1. Find the sum of 10 terms of the series 1, 2, 4, &c.

Here $$a=1,\ r=2,\ n=10.$$

$$\therefore\ S=a\frac{1-r^n}{1-r}$$

$$=\frac{1-2^{10}}{1-2}=2^{10}-1=1023.$$

Ex. 2. Find the sum of 6 terms of the series

$$2-3+\frac{9}{2}-\&c.$$

Here $a=2$, $r=\frac{-3}{2}$, and $n=6$.

$$\therefore\ S=a\frac{1-r^n}{1-r}$$

$$=2\frac{1-(-\frac{3}{2})^6}{1-(-\frac{3}{2})}=2\frac{1-\frac{3^6}{2^6}}{1+\frac{3}{2}}$$

$$=\tfrac{4}{5}\left(1-\frac{3^6}{2^6}\right)=-8\tfrac{5}{16}.$$

223. **Sum of an infinite number of terms of a geometrical progression.** It might at first be thought that, by taking a very great number of terms of any geometrical progression, the sum could be made as great as we please; an example will, however, shew that this is not the case. Suppose that we have a line two inches long, and divide it into two equal parts and take away one; then bisect the remainder and take away one of the parts; and suppose this process continued to any extent. Then the lengths in inches of the successive parts taken will be 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. Now it is clear that the sum of all the parts taken away can never *exceed* two inches, but, as the part which is left over will diminish without limit as the number of the operations is increased, the sum of the parts taken away

can be made to differ from two inches by a length which is smaller than any conceivable length; hence the sum of

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \dots$$

continued to infinity is equal to 2.

From Art. 222 we have

$$S = \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

Now if r is a *proper fraction*, whether positive or negative, the absolute value of r^n will decrease as n increases; moreover r^n can be made *as small as we please* by sufficiently increasing the value of n.

Hence, *when r is numerically less than unity*, the sum of the series can be made to differ from $\dfrac{a}{1 - r}$ by *as small a quantity as we please* by taking a very great number of terms.

Thus the sum of an *infinite number* of terms of the geometrical progression $a + ar + ar^2 + \dots$, in which *r is numerically less than unity*, is $\dfrac{a}{1 - r}$.

Ex. 1. Find the sum of an infinite number of terms of the series

(i) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

(ii) $9 - 6 + 4 - \dots$

In (i) $$a = 1, \quad r = \tfrac{1}{2};$$

$$\therefore S = \frac{a}{1 - r} = \frac{1}{1 - \frac{1}{2}} = 2.$$

In (ii) $$a = 9, \quad r = \frac{-6}{9} = -\frac{2}{3};$$

$$\therefore S = \frac{a}{1 - r} = \frac{9}{1 - \left(-\dfrac{2}{3}\right)} = \frac{9}{\frac{5}{3}} = \frac{27}{5}.$$

Ex. 2. Find the value of $\cdot 23\dot{4}$.

$$\cdot 23\dot{4} = \cdot 23444\ldots\ldots\ldots$$

$$= \frac{2}{10} + \frac{3}{10^2} + \frac{4}{10^3} + \frac{4}{10^4} + \frac{4}{10^5} + \&\text{c.}$$

Now $\dfrac{4}{10^3} + \dfrac{4}{10^4} + \ldots$ to infinity $= \dfrac{\dfrac{4}{10^3}}{1 - \dfrac{1}{10}} = \dfrac{10}{9} \times \dfrac{4}{10^3} = \dfrac{4}{900}$.

Hence $\cdot 23\dot{4} = \dfrac{2}{10} + \dfrac{3}{100} + \dfrac{4}{900} = \dfrac{211}{900}$.

Ex. 3. Find the geometrical progression whose sum to infinity is 18, and whose second term is -8.

Let a be the first term, and r be the common ratio.

Then the second term is ar, and the sum to infinity is $\dfrac{a}{1-r}$.

Hence $ar = -8$,

and $\dfrac{a}{1-r} = 18$.

Hence, by division, we have

$$r(1-r) = \frac{-8}{18};$$

$$\therefore\ r^2 - r = \frac{4}{9};$$

$$\therefore\ r = -\frac{1}{3},\ \text{or}\ r = \frac{4}{3}.$$

If $r = -\dfrac{1}{3}$, $a = \dfrac{-8}{r} = 24$.

Thus the series is 24, -8, $\dfrac{8}{3}$, &c.

The value $r = \dfrac{4}{3}$ is inadmissible, for the sum of a geometrical progression is not given by the formula

$$S = \frac{a}{1-r},$$

except when r is *numerically less than unity*.

EXAMPLES. LXVI.

Find the sum of the following series:

1. $8+12+18+27+\dots$ to n terms. **2.** $12+9+6\frac{3}{4}+\dots$ to 6 terms.

3. $a+\frac{a}{r}+\frac{a}{r^2}+\dots$ to n terms. **4.** $1\frac{1}{2}+1\frac{1}{3}+1\frac{5}{27}+\dots$ to 8 terms.

5. $\frac{2}{9}+\frac{1}{3}+\frac{1}{2}+\dots$ to 6 terms. **6.** $12-9+6\frac{3}{4}-\dots$ to infinity.

7. $4+1{\cdot}2+{\cdot}36+{\cdot}108+\dots$ to infinity.

8. $4+{\cdot}8+{\cdot}16+\dots$ to infinity. **9.** $a^2+ab+b^2+\dots$ to n terms.

10. $3-2+\frac{4}{3}-\dots$ to infinity.

11. Find the sum of the successive powers of 2 from 2 up to 4096 inclusive.

12. Find, correct to four places of decimals, the sum to infinity of the series $1+\frac{1}{\sqrt{2}}+\frac{1}{2}+\dots$

13. Find the geometric mean of $4x^2-12x+9$ and $9x^2+12x+4$.

14. Shew that the product of any odd number of terms of a G.P. is equal to the nth power of the middle term, n being the number of the terms.

15. Find the G.P. whose sum to infinity is 4, and whose second term is $\frac{3}{4}$.

16. Find the G.P. whose sum to infinity is 9, and whose second term is -4.

17. The sum of the first 10 terms of a certain G.P. is equal to 33 times the sum of the first 5 terms. What is the common ratio?

18. If the sum of a geometrical series to infinity is n times the first term, shew that the common ratio is $1-\frac{1}{n}$.

19. If the common ratio of successive terms of a G.P. be positive and less than $\frac{1}{2}$, shew that each term is greater than the sum of all that follow it.

20. The sixth term of a G.P. is 8 times the third term, and the sum of the first two terms is 24; find the series.

21. Find the sum to infinity of the series

$$\frac{a}{a+b}+\left(\frac{a}{a+b}\right)^2+\left(\frac{a}{a+b}\right)^3+\ldots$$

22. Find the sum to infinity of the series

$$1+\left(1-\frac{1}{a}\right)+\left(1-\frac{1}{a}\right)^2+\left(1-\frac{1}{a}\right)^3+\ldots$$

23. Shew that, if

$$S_1=a+ar+ar^2+ar^3+\ldots+ar^{2n},$$

and $$S_2=a-ar+ar^2-ar^3+\ldots+ar^{2n};$$

then $$S_1S_2=a^2+a^2r^2+a^2r^4+\ldots+a^2r^{4n}.$$

24. Sum the series $\sqrt{\frac{3}{2}}+\frac{1}{3}\sqrt{2}+\frac{2}{9}\sqrt{\frac{2}{3}}+\ldots$ to infinity.

25. The sixth term of a G.P. is $\frac{1}{33614}$ and the second term is $\frac{1}{14}$; find the sum to infinity.

26. Find three numbers in G.P. such that their sum is 14, and the sum of their squares is 84.

27. The sum of the first, second and third terms of a G.P. is 42, and the fourth term exceeds the first by 126. What is the series?

28. The sum of the first, second and third terms of a G.P. is to the sum of the third, fourth and fifth terms as 1 : 4, and the fifth term is 8. What is the series?

29. Shew that, if a, b, c, d be in G.P., then will $a+b$, $b+c$, $c+d$, and also a^2+b^2, b^2+c^2, c^2+d^2 be in G.P.

30. Shew that, if a, b, c be the pth, qth, and rth terms respectively of a G.P., then will $a^{q-r}\,b^{r-p}\,c^{p-q}=1$.

31. Shew that, if $(a^2+b^2)(b^2+c^2)=(ab+bc)^2$, then will a, b, c be in G.P.

224. Sometimes we are not told the law which connects successive terms of a series; but when a certain number of the terms are given, the law can in simple cases be at once determined.

Suppose, for example, we have the series

3, 9, 15, &c.

Here $9-3=6$, and $15-9=6$.

Thus the series is an *arithmetical progression*, whose common difference is 6.

Again, in the series

$$3,\ 9,\ 27,\ \&c.$$

Since $9-3=6$, and $27-9=18$, the series is *not* an arithmetical progression. We then see whether it satisfies the conditions for being a geometrical progression, namely $\frac{9}{3}=\frac{27}{9}$, which is the case. Thus the series is a *geometrical progression*, whose common ratio is 3.

EXAMPLES. LXVII.

Sum the following series :

1. $3,\ 2{\cdot}7,\ 2{\cdot}4$, &c. to 21 terms. **2.** $2,\ 18,\ 162$, &c. to 7 terms.

3. $1,\ {\cdot}2,\ {\cdot}04$, &c. to infinity. **4.** $a,\ b,\ \frac{b^2}{a}$, &c. to n terms.

5. ${\cdot}3,\ {\cdot}03,\ {\cdot}003$, &c. to infinity. **6.** $3+4{\cdot}3+5{\cdot}6+\ldots$ to 11 terms.

7. $a+\frac{4a+b}{3}+\frac{5a+2b}{3}+\ldots$ to 19 terms.

8. $\frac{1}{4}-\frac{1}{3}+\frac{4}{9}-$ &c. to 8 terms. **9.** $3\frac{1}{8}+1\frac{1}{4}+\frac{1}{2}+$ &c. to infinity.

10. $2\frac{1}{2}+6\frac{1}{4}+15\frac{5}{8}+\ldots$ to 5 terms.

11. Sum the following series to 6 terms, and when possible to infinity:

(i) $2\frac{1}{2}+6\frac{1}{4}+10+\ldots$ (ii) $9+5+1+\ldots$ (iii) $4-3+\frac{9}{4}-\ldots$

(iv) $\frac{1}{3}+\frac{1}{2}+\frac{3}{4}+\ldots$ (v) $\frac{1}{2}+\frac{1}{3}+\frac{2}{9}+\ldots$

12. Shew that, if an odd number of quantities are in geometrical progression, the first, the middle and the last of them are also in geometrical progression.

13. The sum of the first 7 terms of an A.P. is 49, and the sum of the next 8 terms is 176: what is the series?

14. The arithmetic mean of the first and third terms of a geometrical progression is 5 times the second term: find the common ratio.

15. If the fourth term in an arithmetical progression is the geometric mean of the second and seventh terms; shew that the sixth term is the geometric mean of the second and fourteenth.

16. If P be the continued product of n terms of a geometrical progression whose first term is a and last term l, shew that

$$P^2 = (al)^n.$$

17. The continued product of three numbers in G.P. is 216, and the sum of the products of them in pairs is 156: find the numbers.

18. Divide 25 into five parts which are in A.P., and which are such that the sum of the squares of the least and greatest of them is one less than the sum of the squares of the other three.

19. Insert between 6 and 16 two numbers such that the first three may be in A.P. and the last three in G.P.

20. If a, b, c be in geometrical progression, and x, y be the arithmetic means between a, b, and b, c respectively, prove that

$$\frac{2}{b} = \frac{1}{x} + \frac{1}{y}, \text{ and } 2 = \frac{a}{x} + \frac{c}{y}.$$

CHAPTER XXIV.

HARMONICAL PROGRESSION. OTHER SIMPLE SERIES.

225. A SERIES of quantities is said to be in **Harmonical Progression** when the difference between the first and the second of any three consecutive terms is to the difference between the second and the third as the first is to the third.

Thus a, b, c, d, &c. are in harmonical progression, if

$$a-b : b-c :: a : c,$$

$$b-c : c-d :: b : d,$$

and so on.

226. If a, b, c are in harmonical progression, we have by definition

$$a-b : b-c :: a : c.$$

Hence $$c(a-b)=a(b-c),$$

or $$ca-bc=ab-ac.$$

Divide by abc; then

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b},$$

which shews that $$\frac{1}{a},\ \frac{1}{b},\ \frac{1}{c}$$

are in arithmetical progression.

Thus, *if quantities are in harmonical progression, their reciprocals are in arithmetical progression.*

The above relation between quantities in harmonical progression is much more frequently employed than that given in the preceding article.

227. When three quantities are in harmonical progression, the middle one is called the **harmonic mean** of the other two.

If a, b, c are in harmonical progression,

$$\frac{1}{a}, \; \frac{1}{b}, \; \frac{1}{c}$$

are in arithmetical progression;

are in arith

$$\frac{2\,a \times b}{a+b}$$

$$\therefore \; \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b};$$

$$\therefore \; \frac{2}{b} = \frac{1}{a} + \frac{1}{c};$$

$$\therefore \; b = \frac{2ac}{a+c}.$$

Thus *the harmonic mean of any two quantities is twice their product divided by their sum.*

228. If we put A, G, H for the arithmetic, the geometric, and the harmonic means respectively of any two quantities a and b, we have

$$A = \tfrac{1}{2}(a+b), \qquad G = \sqrt{ab}, \qquad \text{and } H = \frac{2ab}{a+b}.$$

Hence $A \times H = \tfrac{1}{2}(a+b) \times \dfrac{2ab}{a+b} = ab;$

$$\therefore \; AH = G^2.$$

Hence G is the geometric mean of A and H.

Thus *the geometric mean of any two quantities is also the geometric mean of their arithmetic and harmonic means.*

$$b = \frac{2ac}{a+c}$$

229. Many questions concerning quantities in harmonical progression can be solved by considering the arithmetical progression whose terms are the reciprocals of the terms of the harmonical progression.

For example, to insert n harmonic means between any two quantities a and b.

Insert n arithmetic means between $\frac{1}{a}$ and $\frac{1}{b}$. These, by Art. 215, will be

$$\frac{1}{a}+\frac{\frac{1}{b}-\frac{1}{a}}{n+1},\quad \frac{1}{a}+2\,\frac{\frac{1}{b}-\frac{1}{a}}{n+1},\ \&\text{c.}$$

Thus $$\frac{1}{a},\quad \frac{1}{a}+\frac{\frac{1}{b}-\frac{1}{a}}{n+1},\quad \frac{1}{a}+2\,\frac{\frac{1}{b}-\frac{1}{a}}{n+1},\ \ldots,\ \frac{1}{b}$$

are in A.P.

That is

$$\frac{1}{a},\quad \frac{(nb+a)}{(n+1)\,ab},\quad \frac{(n-1)\,b+2a}{(n+1)\,ab},\ \ldots\ldots,\ \frac{1}{b}$$

are in A.P.

Hence the reciprocals of these are in H.P.

Thus the required harmonic means are

$$\frac{(n+1)\,ab}{nb+a},\quad \frac{(n+1)\,ab}{(n-1)\,b+2a},\ \&\text{c.}$$

NOTE. It is of importance to notice that no formula can be found giving the sum of any number of terms in harmonical progression.

230. The conditions that a, b, c may be in A.P., G.P., or H.P. respectively may be written

$$a-b : b-c :: a : a,$$
$$a-b : b-c :: a : b,$$
$$a-b : b-c :: a : c.$$

The first and third follow from the definitions of A.P. and H.P., and the second can be easily verified by multiplication.

Ex. 1. The second term of a harmonical progression is 2, and the fourth term is 6: find the series.

The second term of the corresponding arithmetical progression is $\frac{1}{2}$, and the fourth term is $\frac{1}{6}$. Hence, if a be the first term, and d the common difference, we have

$$a+d=\tfrac{1}{2}, \text{ and } a+3d=\tfrac{1}{6},$$

whence $$a=\tfrac{2}{3}, \text{ and } d=-\tfrac{1}{6}.$$

Thus the arithmetical progression is

$$\tfrac{2}{3},\quad \tfrac{1}{2},\quad \tfrac{1}{3},\quad \tfrac{1}{6}, \text{ \&c.}$$

Hence the harmonical progression is

$$\tfrac{3}{2},\quad 2,\quad 3,\quad 6, \text{ \&c.}$$

Ex. 2. Shew that $a+b$, $2b$ and $b+c$ will be in H.P. if a, b, c are in G.P.

The three quantities $a+b$, $2b$ and $b+c$ will be in H.P. if

$$\frac{1}{a+b}+\frac{1}{b+c}=\frac{2}{2b},$$

or if $$b(b+c)+b(a+b)=(a+b)(b+c);$$

that is, if $$b^2+bc+ab+b^2=ab+b^2+ac+bc,$$

or if $$b^2=ac;$$

and this is the case when a, b, c are in G.P.

EXAMPLES. LXVIII.

1. Shew that if the terms of a harmonic progression be all multiplied by the same quantity, the products will be in harmonical progression.

2. Insert 5 harmonic means between 1 and 7.

3. Insert 4 harmonic means between $\frac{2}{3}$ and $\frac{2}{13}$.

4. Shew that, if the arithmetic mean of two quantities is 1, their harmonic mean will be the square of their geometric mean.

5. Shew that, if a^2, b^2, c^2 are in arithmetic progression, $b+c$, $c+a$ and $a+b$ will be in harmonical progression.

6. Shew that, if x, y, z be in H.P.,

$$(y+z-x)^2,\ (z+x-y)^2,\ (x+y-z)^2$$

will be in A.P.

7. Shew that, if x, y, z be in H.P., then will $\dfrac{x}{y+z}$, $\dfrac{y}{z+x}$, $\dfrac{z}{x+y}$ be also in H.P.

8. Shew that, if x, y, z be in H.P., then will

$$\frac{x}{y+z-x},\ \frac{y}{z+x-y},\ \frac{z}{x+y-z}$$

be also in H.P.

9. If a, b, c, d be in H.P., then will $\dfrac{a}{b+c+d}$, $\dfrac{b}{c+d+a}$, $\dfrac{c}{d+a+b}$ and $\dfrac{d}{a+b+c}$ be in H.P.

10. If a, b, c be in H.P., then will $2a-b$, b and $2c-b$ be in G.P.

11. If a, b, c be in H.P., then will a, $a-c$ and $a-b$ be in H.P.

12. Shew that

$$(a^2+b^2)(a^2+ab+b^2),\ a^4+a^2b^2+b^4, \text{ and } (a^2+b^2)(a^2-ab+b^2)$$

are in H.P.

13. Shew that, if x, a_1, a_2, y be in A.P., x, g_1, g_2, y in G.P., and x, h_1, h_2, y in H.P.; then will

$$\frac{g_1g_2}{h_1h_2}=\frac{a_1+a_2}{h_1+h_2}.$$

14. If a, b, c be in H.P., then will

$$\frac{b+a}{b-a}+\frac{b+c}{b-c}=2.$$

15. If a, b, c be in G.P., then will $\dfrac{2}{a+b}$, $\dfrac{1}{b}$ and $\dfrac{2}{b+c}$ be in A.P.

16. If a, b, c be in A.P., and b, c, d in H.P., then will $a : b = c : d$.

17. Shew that if

$$\frac{1}{a}+\frac{1}{c}=\frac{1}{b-a}+\frac{1}{b-c};$$

then $b=a+c$, or else a, b, c are in harmonical progression.

18. If a, b, c, d be in H.P., then will

$$3(b-a)(d-c)=(c-b)(d-a).$$

19. Shew that, if a, b, c be in H.P., then

$$\frac{2}{b}=\frac{1}{b-a}+\frac{1}{b-c}.$$

20. Three numbers are in A.P. The product of the extremes is five times the mean, and the sum of the two largest is 3 times the least. Find the numbers.

21. Shew that, if $\frac{a+b}{1-ab}$, b, $\frac{b+c}{1-bc}$ are in A.P., then a, $\frac{1}{b}$, c will be in H.P.

22. If a, b, c be in A.P., and a^2, b^2, c^2 in H.P., prove that $-\frac{a}{2}$, b, c are in G.P. or $a=b=c$.

23. Three numbers are in A.P., and if 1 be added to the greatest they will be in G.P., shew that the smallest is equal to the square of the common difference.

24. Shew that, if x, a, b, y be in A.P. and x, u, v, y in H.P.; then

$$av=bu=xy.$$

25. Shew that, if a_1, a_2, a_3, a_4 be in H.P., then

$$a_1a_2+a_2a_3+a_3a_4=3a_1a_4.$$

26. Shew that, if a, b, c be in A.P., and a, c, b be in G.P., then will b, a, c be in H.P.

231. **Other simple series.** There are, besides the progressions, many other series the successive terms of which are formed according to simple laws. The following are examples of such series:

$$1^2+2^2+3^2+\ldots\ldots+n^2,$$

$$1\,.\,2+2\,.\,3+3\,.\,4+\ldots\ldots+n(n+1),$$

and

$$\frac{1}{1\,.\,2}+\frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}+\ldots\ldots+\frac{1}{n(n+1)};$$

232. We proceed to find the sum of n terms of some of these simple series.

The sum of n terms of a series is generally denoted by S_n, and the sum of the series when continued without limit by S_∞.

Ex. 1. Find the sum of n terms of the series

$$1\,.\,2+2\,.\,3+3\,.\,4+4\,.\,5+\ldots.$$

Here

$$S_n=1\,.\,2+2\,.\,3+3\,.\,4+\ldots+(n-1)\,n+n\,(n+1).$$

Let

$$\Sigma=1\,.\,2\,.\,3+2\,.\,3\,.\,4+3\,.\,4\,.\,5+\ldots+(n-1)n(n+1)+n(n+1)(n+2).$$

Shifting each term one place to the right we have

$$\Sigma= \qquad 1\,.\,2\,.\,3+2\,.\,3\,.\,4+\ldots+(n-2)(n-1)n+(n-1)\,n\,(n+1) +n(n+1)(n+2).$$

Now subtract the last result from the preceding, taking each term from the one above it; then we have

$$0=1\,.\,2\,.\,3+3\,.\,2\,.\,3+3\,.\,3\,.\,4+\ldots+3\,(n-1)\,n \quad +3n(n+1) -n\,(n+1)(n+2).$$

Hence

$$3\,\{1\,.\,2+2\,.\,3+3\,.\,4+\ldots+(n-1)\,n+n\,(n+1)\}=n\,(n+1)\,(n+2)\,;$$

$$\therefore\ S_n=\tfrac{1}{3}n\,(n+1)\,(n+2).$$

Ex. 2. Find the sum of n terms of the series

$$1\,.\,2\,.\,3+2\,.\,3\,.\,4+3\,.\,4\,.\,5+\ldots.$$

Here

$$S_n=1\,.\,2\,.\,3+2\,.\,3\,.\,4+3\,.\,4\,.\,5+\ldots+(n-1)n(n+1)+n(n+1)\,(n+2).$$

Let

$$\Sigma=1\,.\,2\,.\,3\,.\,4+2\,.\,3\,.\,4\,.\,5+3\,.\,4\,.\,5\,.\,6+\ldots +(n-1)\,n\,(n+1)\,(n+2)+n\,(n+1)\,(n+2)\,(n+3).$$

Then

$$\Sigma= \qquad 1\,.\,2\,.\,3\,.\,4+2\,.\,3\,.\,4\,.\,5+\ldots +(n-2)\,(n-1)\,n\,(n+1)+(n-1)\,n\,(n+1)\,(n+2) +n\,(n+1)\,(n+2)\,(n+3).$$

Hence, by subtraction, each term being taken from the one above it,

$$0=1\,.\,2\,.\,3\,.\,4+4\,.\,2\,.\,3\,.\,4+4\,.\,3\,.\,4\,.\,5+\ldots +4\,(n-1)\,n\,(n+1)+4\,n\,(n+1)\,(n+2) -n\,(n+1)\,(n+2)\,(n+3);$$

$$\therefore\ 4\{1.2.3+2.3.4+3.4.5+\ldots+n(n+1)(n+2)\}$$
$$=n(n+1)(n+2)(n+3);$$
$$\therefore\ S_n=\tfrac{1}{4}n(n+1)(n+2)(n+3).$$

NOTE. This series and the preceding are examples of an important type of series in which (i) each term contains the same number of factors, (ii) the factors of any term are in A.P., and (iii) the first factors of the successive terms form the same A.P. as the successive factors of the first term.

The Σ series by means of which the required sum can always be found is the series whose terms are formed according to the same law but with an *additional factor at the end.*

Ex. 3. To find the sum of n terms of the series

$$1^2+2^2+3^2+\ldots$$

Here $S_n=1^2+2^2+3^2+\ldots+n^2.$

Now $n^2=n(n+1)-n;$

$$\therefore\ S_n=1.2+2.3+3.4+\ldots+n(n+1)$$
$$-1-2-3-\ldots-n.$$

Now, by Example 1,

$$1.2+2.3+3.4+\ldots+n(n+1)=\tfrac{1}{3}n(n+1)(n+2);$$

also $1+2+3+\ldots+n=\tfrac{1}{2}n(n+1).$

Hence
$$S_n=\tfrac{1}{3}n(n+1)(n+2)-\tfrac{1}{2}n(n+1)$$
$$=\tfrac{1}{6}n(n+1)(2n+1).$$

Ex. 4. To find the sum of n terms of the series

$$1^3+2^3+3^3+\ldots+n^3.$$

Here $S_n=1^3+2^3+3^3+\ldots+n^3.$

Now $4n^3=\{n(n+1)\}^2-\{(n-1)n\}^2$, for all values of n.

Hence we have

$$\begin{aligned}
4.1^3&=(1.2)^2,\\
4.2^3&=(2.3)^2-(1.2)^2,\\
4.3^3&=(3.4)^2-(2.3)^2,\\
\ldots\ldots&=\ldots\ldots\ldots\ldots\ldots\ldots\\
4(n-1)^3&=\{(n-1)n\}^2-\{(n-2)(n-1)\}^2,\\
4n^3&=\{n(n+1)\}^2-\{(n-1)n\}^2.
\end{aligned}$$

Hence by addition

$$4\{1^3+2^3+3^3+\ldots+n^3\}=\{n(n+1)\}^2;$$

$$\therefore\ 1^3+2^3+3^3+\ldots+n^3=\tfrac{1}{4}n^2(n+1)^2.$$

This result may be expressed differently; for since

$$1+2+3+\ldots+n=\tfrac{1}{2}n(n+1),$$

we have $\quad 1^3+2^3+3^3+\ldots+n^3=(1+2+3+\ldots+n)^2.$

Thus *the sum of the cubes of the first* n *numbers is equal to the square of their sum.*

Ex. 5. Find the sum of n terms of the series

$$\frac{1}{1\,.\,2}+\frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}+\ldots$$

Here

$$S_n=\frac{1}{1\,.\,2}+\frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}+\ldots+\frac{1}{(n-1)\,n}+\frac{1}{n\,(n+1)}.$$

Let $\quad \Sigma=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}+\frac{1}{n+1},$

then $\quad \Sigma=\quad\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}+\frac{1}{n}+\frac{1}{n+1}.$

Hence, by subtraction,

$$0=1-\frac{1}{1\,.\,2}-\frac{1}{2\,.\,3}-\frac{1}{3\,.\,4}-\ldots-\frac{1}{(n-1)\,n}-\frac{1}{n\,(n+1)}-\frac{1}{n+1}.$$

Hence $\quad \frac{1}{1\,.\,2}+\frac{1}{2\,.\,3}+\ldots+\frac{1}{n\,(n+1)}=1-\frac{1}{n+1}.$

When n is infinite $\frac{1}{n+1}$ is zero; hence the sum of an infinite number of terms of the series $\frac{1}{1\,.\,2}+\frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}+\ldots$ is 1.

Ex. 6. Find the sum of n terms of the series

$$\frac{1}{1\,.\,2\,.\,3}+\frac{1}{2\,.\,3\,.\,4}+\frac{1}{3\,.\,4\,.\,5}+\ldots$$

Take a series Σ formed according to the same law but with one factor less in each term, and proceed as before. The result is $\frac{1}{2}\left\{\frac{1}{1\,.\,2}-\frac{1}{(n+1)(n+2)}\right\}.$

Ex. 7. Find the sum of n terms of the series

$$1+2x+3x^2+4x^3+\dots$$

Let $S=1+2x+3x^2+4x^3+\dots+nx^{n-1}+(n+1)\,x^n$.

Then $Sx=\quad x+2x^2+3x^3+\dots+\;nx^n\;+(n+1)\,x^{n+1}$.

Hence, by subtraction,

$$S\,(1-x)=1+x+x^2+x^3+\dots+x^n-(n+1)\,x^{n+1}.$$

But $$1+x+x^2+\dots+x^n=\frac{1-x^{n+1}}{1-x}.$$

Hence $$S\,(1-x)=\frac{1-x^{n+1}}{1-x}-(n+1)\,x^{n+1};$$

$$\therefore\; S=\frac{1-x^{n+1}}{(1-x)^2}-(n+1)\,\frac{x^{n+1}}{1-x}.$$

EXAMPLES. LXIX.

Find the sum of the following series to n terms, and when possible to infinity.

1. $2\,.\,4+4\,.\,6+6\,.\,8+\dots$ **2.** $3\,.\,5+5\,.\,7+7\,.\,9+\dots$

3. $1\,.\,4+4\,.\,7+7\,.\,10+\dots$ **4.** $2\,.\,5+5\,.\,8+8\,.\,11+\dots$

5. $1\,.\,3\,.\,5+3\,.\,5\,.\,7+5\,.\,7\,.\,9+\dots$

6. $2\,.\,7\,.\,12+7\,.\,12\,.\,17+12\,.\,17\,.\,22+\dots$

7. $\dfrac{1}{1\,.\,3}+\dfrac{1}{3\,.\,5}+\dfrac{1}{5\,.\,7}+\dots$ **8.** $\dfrac{1}{2\,.\,5}+\dfrac{1}{5\,.\,8}+\dfrac{1}{8\,.\,11}+\dots$

9. $\dfrac{1}{1\,.\,3\,.\,5}+\dfrac{1}{3\,.\,5\,.\,7}+\dfrac{1}{5\,.\,7\,.\,9}+\dots$

10. $\dfrac{1}{2\,.\,5\,.\,8}+\dfrac{1}{5\,.\,8\,.\,11}+\dfrac{1}{8\,.\,11\,.\,14}+\dots$

11. $\dfrac{1}{(x+1)\,(x+2)}+\dfrac{1}{(x+2)\,(x+3)}+\dfrac{1}{(x+3)\,(x+4)}+\dots$

12. $\dfrac{1}{(1+x)\,(1+2x)}+\dfrac{1}{(1+2x)\,(1+3x)}+\dfrac{1}{(1+3x)\,(1+4x)}+\dots$

13. $\dfrac{1}{1}+\dfrac{1}{1+2}+\dfrac{1}{1+2+3}+\dfrac{1}{1+2+3+4}+\dots$

14. $1^2+3^2+5^2+\dots$ **15.** $1^3+3^3+5^3+\dots$

16. $a\,(a+b)+(a+b)\,(a+2b)+(a+2b)\,(a+3b)+\dots$

MISCELLANEOUS EXAMPLES. VI.

A. 1. Simplify

$$(y+3)(y^2-1)-3(y+1)(y^2-9)+3(y-1)(y^2-9)-(y-3)(y^2-1).$$

2. Prove that

$$\frac{(2y-x)^3-(2x-y)^3}{3(y-x)}+\frac{(2y-x)^3+(2x-y)^3}{x+y}=10x^2-16xy+10y^2.$$

3. Find the H.C.F. of $3x^3-13x^2+23x-21$ and $6x^3+x^2-44x+21$. For what value of x will both expressions vanish? 4

4. Solve the equations:

(i) $3x+\dfrac{2}{y}-1=12x+\dfrac{5}{y}+14=\dfrac{1}{y}-2x-14.$

(ii) $x^2-(a+b+2c)x+(a+b+c)c=0.$

5. A man buys 9 oxen and 20 sheep for £230; by selling the oxen at a gain of 25 per cent., and the sheep at a loss of 20 per cent., he gains £35 altogether. Find the price he gave for each.

6. If α and β be the roots of the equation $x^2+4x+3=0$, shew that the equation whose roots are $\dfrac{\alpha+\beta}{\alpha}$ and $\dfrac{\alpha+\beta}{\beta}$ is $3x^2-16x+16=0$.

7. Multiply $a^{\frac{2}{3}}b^{-1}-2a^{\frac{1}{3}}b^{-\frac{1}{2}}+4-8a^{-\frac{1}{3}}b^{\frac{1}{2}}+16a^{-\frac{2}{3}}b$ by $a^{\frac{1}{3}}+2b^{\frac{1}{2}}$.

8. Rationalise the denominator of $\dfrac{5+4\sqrt{3}}{5-4\sqrt{3}}$,

and simplify $\sqrt{25+4\sqrt{34}}$.

9. Sum the series $5-3\frac{3}{4}+2\frac{13}{16}-$ &c. to 8 terms, and find its limit when continued to infinity.

10. Insert 20 arithmetic means between 100 and 300, and find their sum.

B. 1. Simplify $3\{x-2(y-z)\}-[4y+2\{x-\overline{y-z}\}]$.

2. Find the factors of

$$mnx^2+m^2xy+n^2xy+mny^2,$$

and of

$$x^3-x^2y+xy^2-y^3.$$

3. If the sum of two numbers be equal to 4, shew that their difference is one-quarter of the difference of their squares.

4. Solve

3 4 (i) $\dfrac{a}{x+a}+\dfrac{b}{x+b}=1$.

5 6 (ii) $\sqrt{12x-3}+\sqrt{x+2}+\sqrt{7x-13}=0$.

5. 7 Shew that $x^2-8x+22$ can never be less than 6.

6. Find the square root of $a^6+2a^3b^2-2a^3c+b^4-2b^2c+c^2$.

7. If $a:b::c:d$, shew that $a^2+b^2:c^2+d^2::(a+b)^2:(c+d)^2$.

8. Sum the following series:

(i) $5-1-7-$&c. to 20 terms.

(ii) $2\frac{1}{2}+1+\frac{2}{5}+\ldots$ to infinity.

9. Find the sum of all the numbers which are less than 1000, and are divisible by 7.

10. A train travelling at the rate of $37\frac{1}{2}$ miles an hour passes a person walking on a road parallel to the railway in 6 seconds; it also meets another person walking at the same rate as the other but in the opposite direction, and passes him in 4 seconds. Find the length of the train.

C. 1. Find the value of $\dfrac{\sqrt{(y^2+z)}}{\sqrt{(z+y)}}-\dfrac{z-x\,(y-z)}{x-z\,(y-x)}$,

when $x=0, \quad y=1, \quad z=1\frac{1}{2}$.

2. Divide $4x^4-9x^2y^2+12xy^3-4y^4$ by $2x^2+3xy-2y^2$.

3. Shew that the square of the sum of two consecutive numbers is equal to four times their product increased by unity.

4. Find the L.C.M. of $x^2-6ax+9a^2$, $x^2-ax-6a^2$ and $3x^2-12a^2$.

5. Simplify $\dfrac{2}{a-2}+\dfrac{1}{1-a}+\dfrac{1}{a+1}-\dfrac{2}{a+2}$.

6. If 40 minutes would be saved in a journey by increasing the rate of the train by 5 miles an hour, and 1 hour would be lost by diminishing it by the same amount, find the rate of the train and the length of the journey.

7. Simplify $2+\sqrt{8}+\sqrt{2}-\sqrt{27}-\sqrt{12}+\sqrt{75}-\sqrt{(19+6\sqrt{2})}$.

8. Prove that any ratio is made nearer to unity by adding the same number to each of its terms. Also shew that $\frac{ma+nc}{mb+nd}$ is intermediate between $\frac{a}{b}$ and $\frac{c}{d}$.

9. Find the sum of the following series:

(i) $21+15+9+\dots$ to 8 terms.

(ii) $5+2+{\cdot}8+\dots$ to infinity.

10. The arithmetic mean of two numbers is 17, and their geometric mean is 15: what are the numbers?

D. 1. Simplify $\{x(x+a)-a(x-a)\}\{x(x-a)-a(a-x)\}$.

2. Shew that $(n+1)^4-n^4=(2n+1)(2n^2+2n+1)$,

and that $(a+b)^4-a^4-b^4=4ab(a+b)^2-2a^2b^2$.

3. Simplify $\left\{1-\frac{3x-20}{x^2-6x}\right\}\left\{1-\frac{8x-42}{x^2-5x}\right\}$.

4. Solve the equations:

(i) $\frac{3}{x}-\frac{5}{y}=21,\quad \frac{5}{x}+\frac{7}{y}=-11$.

(ii) $3x^2-4xy+2y^2=33,\quad x^2-y^2=16$.

5. A number consists of two digits, of which that in the unit's place is the greater; the difference between the squares of the digits is equal to the number, and if the digits were inverted the number thus formed would be 7 times the sum of the digits. Find the number.

6. Find the equation whose roots are the cubes of the roots of the equation $x^2-4x+2=0$.

7. Find the square root of $x^6-2x^2+8+x^{-2}-8x^{-4}+16x^{-6}$.

8. If $a:b::c:d$, prove that $\frac{a^2+b^2}{c^2+d^2}=\left(\frac{a-b}{c-d}\right)^2$.

9. Sum the following series:

(i) $\frac{1}{3}-\frac{1}{2}+\frac{3}{4}-\dots$ to 10 terms.

(ii) $9-6+4-\dots$ to infinity.

10. The series of odd numbers is divided into groups as follows: 1; 3, 5; 7, 9, 11; 13, 15, 17, 19; and so on. Shew that the sum of the numbers in the nth group is n^3.

E. 1. Simplify $x^2-2xy+y^2-(x^2+xy+y^2)-\{x(-2y+x)+y^2\}$.

2. Divide $6a^4+4b^4-a^3b+13ab^3+2a^2b^2$ by $2a^2+4b^2-3ab$.

3. Find the highest common divisor and the lowest common multiple of $3a^3-3a^2b+ab^2-b^3$ and $4a^2-5ab+b^2$.

4. Simplify $$\frac{\dfrac{4}{x-1}+\dfrac{x-1}{x}}{\dfrac{1}{x-1}-\dfrac{1}{x}}.$$

5. If α and β are roots of $x^2+mx+n=0$, find $\alpha^2\beta+\beta^2\alpha$ in terms of m and n.

Test your result on the equation $x^2-3x+2=0$.

6. Find the value of x for which $3x^2+5x+3$ has its least possible value, and shew that the least value is $\frac{11}{12}$.

7. Shew that, if a, b, c, d be in continued proportion, $$\left(\frac{a-b}{b-c}\right)^3=\frac{a}{d}.$$

8. Find the square root of $$4x^{\frac{4}{3}}-12xy^{\frac{1}{2}}-7x^{\frac{2}{3}}y+24x^{\frac{1}{3}}y^{\frac{3}{2}}+16y^2.$$

9. Sum the series:

(i) $-3-2-1\ldots$ to n terms.

(ii) $1-\frac{1}{2}+\frac{1}{4}\ldots$ to infinity.

10. A train 72 yards long passed another train 60 yards long, which was going in the opposite direction on a parallel line of rails, in 4 seconds. Had the first-mentioned train been travelling at twice its actual speed, the trains would have passed each other in 3 seconds. Find the number of miles per hour at which the trains were travelling.

F. 1. Simplify $(x+y+z)^2-(-x+y+z)^2+(x-y+z)^2-(x+y-z)^2$.

2. Divide $a^3-3a^2b+3ab^2-b^3-c^3$ by $a-b-c$.

3. Prove that, if $$x=a+d, \qquad y=b+d, \qquad z=c+d,$$ then $$x^2+y^2+z^2-yz-zx-xy=a^2+b^2+c^2-bc-ca-ab.$$

4. Simplify $$\frac{1}{a-2x}+\frac{2a}{4x^2-a^2}-\frac{1}{a+2x}.$$

5. Solve the equations:

(i) $$\left.\begin{aligned} 2x+4y-3z&=22 \\ 4x-2y+5z&=18 \\ 5x+\ y-2z&=14 \end{aligned}\right\}.$$

(ii) $$\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}.$$

(iii) $$\left.\begin{aligned} \frac{x+y}{x-y}+\frac{x-y}{x+y}&=\frac{5}{2} \\ x^2+y^2&=90 \end{aligned}\right\}.$$

6. Shew that, if any integer be put for x in the expression

$$x^6-4x^5+14x^4-32x^3+49x^2-60x+36,$$

the result will be a square number.

7. Reduce $\dfrac{1}{\sqrt{(12-\sqrt{140})}}$ to the best form for numerical calculations, and find its value to 4 places of decimals.

8. Shew that the ratio $a-x : a+x$ is greater or less than the ratio $a^2-x^2 : a^2+x^2$, according as the ratio $x : a$ is greater or less than unity.

9. If $\dfrac{1}{x}+\dfrac{1}{y}$ varies inversely as $x+y$, then x^2+y^2 varies as xy.

10. If a, b and c are the sums of n, $2n$ and $3n$ terms respectively of a geometric progression, then $a^2+b^2=a(b+c)$.

CHAPTER XXV.

PERMUTATIONS AND COMBINATIONS.

233. **Definition.** The different ways in which r things can be taken from n things, regard being had to the order of selection or arrangement, are called the **permutations** of the n things r at a time.

Thus two permutations will be different unless they contain the same objects arranged in the same order.

For example, suppose there are four objects, represented by the four letters a, b, c, d.

If we take the four objects one at a time, we have the four permutations a, b, c, d.

If we take the objects two at a time we have the twelve permutations

$$ab,\ ac,\ ad,\ ba,\ bc,\ bd,\ ca,\ cb,\ cd,\ da,\ db,\ dc.$$

Now to find the whole number of the permutations three at a time we can proceed thus. Take any one of the permutations two at a time and place after it either of the two letters which it does not contain, then all the permutations so obtained will be different, for either we shall have used a different permutation of two letters or else the final letters will be different; moreover, since all the possible permutations two together have been used, no permutation of three letters can have been omitted. Hence the number of permutations of 4 things 3 together = number of permutations of 4 things 2 together $\times 2 = 12 \times 2 = 24$.

We can by similar reasoning find the permutations of n things r together.

The number of permutations of n different things taken r at a time is denoted by the symbol ${}_nP_r$.

234. *To find the number of permutations of* n *different things taken* r *at a time, where* r *is any integer not greater than* n.

Let the different things be represented by the letters a, b, c, ...

It is obvious that there are n permutations of the n things when taken one at a time, so that ${}_nP_1 = n$.

Now, if we take any one of the different permutations $(r-1)$ at a time, and place after it any one of the $n-(r-1)$ letters which it does not contain, we shall obtain a permutation of the n things r at a time. We thus obtain

$$n-(r-1)=n-r+1$$

different permutations r at a time from every one of the different permutations $(r-1)$ at a time.

Hence $${}_nP_r = {}_nP_{r-1} \times (n-r+1).$$

Since the above relation is true for all values of r, we have in succession

$$\begin{aligned}
{}_nP_{r-1} &= {}_nP_{r-2} \times (n-r+2),\\
{}_nP_{r-2} &= {}_nP_{r-3} \times (n-r+3),\\
\ldots &= \ldots\\
{}_nP_3 &= {}_nP_2 \times (n-2),\\
{}_nP_2 &= {}_nP_1 \times (n-1).
\end{aligned}$$

Also $${}_nP_1 = \quad n.$$

Multiply all the corresponding members of the above equalities, and cancel all the common factors; then we have

$${}_nP_r = n(n-1)(n-2)\ldots(n-r+1),$$

r being the number of factors on the right.

If all the n things are to be taken r is equal to n, and we have

$$_nP_n = n(n-1)(n-2)\ldots 3 \, . \, 2 \, . \, 1.$$

Def. The continued product $n(n-1)(n-2)\ldots 3 \, . \, 2 \, . \, 1$ is denoted by the symbol $\underline{|n}$, or by $n\,!$ The symbols $\underline{|n}$ and $n!$ are read 'factorial n.' Thus $\underline{|4} = 4 \, . \, 3 \, . \, 2 \, . \, 1.$

Ex. 1. In how many different ways can 6 boys stand in a row?

The number required $= {}_6P_6 = 6 \, . \, 5 \, . \, 4 \, . \, 3 \, . \, 2 \, . \, 1 = 720.$

Ex. 2. How many different numbers can be formed by using three of the figures 1, 2, 3, 4, 5?

The number required $= {}_5P_3 = 5 \, . \, 4 \, . \, 3 = 60.$

Ex. 3. Shew that ${}_{10}P_4 = {}_7P_7$,

${}_{10}P_4 = 10 \, . \, 9 \, . \, 8 \, . \, 7$ and ${}_7P_7 = 7 \, . \, 6 \, . \, 5 \, . \, 4 \, . \, 3 \, . \, 2 \, . \, 1.$

235. *To find the number of permutations of* n *things taken all together, when the things are not all different.*

Let there be n letters; and suppose p of them to be a's, q of them to be b's, r of them to be c's, and so on.

Let P be the required number of permutations.

If in any one of the actual permutations, we suppose that the a's are all changed into p letters different from each other and from all the rest; then, by changing only the arrangement of these p new letters, we should, instead of a single permutation, have $\underline{|p}$ different permutations.

Hence, if the a's were all changed into p letters different from each other and from all the rest, the b's, c's &c. being unaltered, there would be $P \times \underline{|p}$ permutations.

Similarly, if in any one of these new permutations we suppose the b's are all changed into q letters different from each other and from all the rest, we should obtain $\underline{|q}$ permutations by changing the arrangement of these q new letters. Hence the whole number of permutations would now be

$$P \times \underline{|p} \times \underline{|q}.$$

By proceeding in this way we see that if all the letters were changed so that no two were alike, the total number of permutations would be

$$P \times \underline{|p} \times \underline{|q} \times \underline{|r} \ldots.$$

But the number of permutations of n different things is $\underline{|n}$.

Hence
$$P \times \underline{|p} \times \underline{|q} \times \underline{|r} \times \ldots = \underline{|n};$$

$$\therefore \quad P = \frac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r} \ldots}.$$

Ex. (i) Find the number of permutations of the six letters *aaabbc* taken all together.

The number $= \dfrac{\underline{|6}}{\underline{|3}\,\underline{|2}\,\underline{|1}} = 60.$

Ex. (ii) How many different numbers can be formed by the figures 2, 3, 3, 4, 4, 4, 4?

The number required $= \dfrac{\underline{|7}}{\underline{|4}\,\underline{|2}} = 105.$

236. **Definition.** The different ways in which a selection of r things can be made from n things, without regard to the order of selection or arrangement, are called the **Combinations** of the n things r at a time.

Thus two combinations will be different unless they both contain precisely the same objects.

For example, suppose there are four objects, represented by the four letters a, b, c, d. The different combinations one at a time are a, b, c, d; the combinations two at a time are ab, ac, ad, bc, bd, cd; the combinations three at a time are abc, abd, acd, bcd; and there is only one combination, namely $abcd$, when all the letters are taken.

The number of combinations of n things r together is denoted by the symbol ${}_nC_r$.

237. *To find the number of combinations of* n *different things taken* r *at a time.*

Since every combination of r different things would give rise to $\underline{|r}$ different permutations, if the order of the letters were altered in every possible way, it follows that

$$_nC_r \times \underline{|r} = {_nP_r} = n(n-1)(n-2)\ldots(n-r+1).$$

Hence $$_nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1 \cdot 2 \cdot 3 \ldots r}.$$

By the following method the number of combinations of n different things r at a time can be found independently of the number of permutations.

Let the different things be represented by the n letters $a, b, c, d, \ldots$.

In the combinations of the n letters r together the number in which a particular letter occurs must be equal to the number of ways in which $r-1$ of the remaining $n-1$ letters may be selected. Hence in the whole number of combinations r together every letter occurs $_{n-1}C_{r-1}$ times, and therefore the total number of letters employed is $_{n-1}C_{r-1} \times n$; but, since there are r letters in each of the $_nC_r$ combinations, the total number of letters employed is $_nC_r \times r$.

Hence $$r \times {_nC_r} = n \times {_{n-1}C_{r-1}}.$$

Since the above relation is true for all values of n and r, we have in succession

$$(r-1) \times {_{n-1}C_{r-1}} = (n-1) \times {_{n-2}C_{r-2}}$$
$$(r-2) \times {_{n-2}C_{r-2}} = (n-2) \times {_{n-3}C_{r-3}}$$
$$\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots$$
$$2 \times {_{n-r+2}C_2} = (n-r+2) \times {_{n-r+1}C_1}.$$

Also $$_{n-r+1}C_1 = n-r+1.$$

Hence, by multiplying corresponding members of the above equalities and cancelling the common factors, we have

$$\underline{|r} \times {}_nC_r = n(n-1)(n-2)\ldots(n-r+1),$$

that is $${}_nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}}.$$

By multiplying the numerator and denominator of the fraction on the right by $\underline{|n-r}$, we have

$${}_nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)\,\underline{|n-r}}{\underline{|r}\,\underline{|n-r}}$$

$$= \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

Ex. 1. In how many ways can 3 boys be selected out of a class of 15?

The number $= {}_{15}C_3 = \dfrac{15 \cdot 14 \cdot 13}{1 \cdot 2 \cdot 3} = 455.$

Ex. 2. There are 6 candidates for 4 vacancies, and every elector can vote for any number of candidates not greater than the number of vacancies. In how many ways is it possible to vote?

An elector can vote for 4 candidates in ${}_6C_4$ ways, for 3 candidates in ${}_6C_3$ ways, for 2 candidates in ${}_6C_2$ ways, and for 1 candidate in ${}_6C_1$ ways. Thus the whole number of ways in which an elector can vote is

$${}_6C_4 + {}_6C_3 + {}_6C_2 + {}_6C_1 = 15 + 20 + 15 + 6 = 56.$$

238. From the formula obtained in the last Article, we have

$${}_nC_{n-r} = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}};$$

$$\therefore\ {}_nC_{n-r} = {}_nC_r,$$

so that *the number of combinations of* n *things* r *together is equal to the number of combinations of* n *things* n−r *together.*

The above proposition follows however at once from the fact that whenever we take r out of n things we must leave $n-r$, and if every set of r things differs in some particular from every other, the corresponding set of $n-r$ things will also differ in some particular from every other. Hence the number of different sets of r things must be equal to the number of different sets of $n-r$ things.

239. *To prove that* ${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$.

The total number of combinations of $(n+1)$ things r together can be divided into two groups according as they do or do not contain a certain particular thing. The number which do not contain that particular thing is the number of ways in which r of the remaining n things can be taken, which is ${}_nC_r$; and the number which do contain the particular thing is the number of ways in which $(r-1)$ of the remaining n things can be taken, which is ${}_nC_{r-1}$. Hence ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$.

The above important proposition may also be proved from the general formula. Thus

$$\begin{aligned}
&{}_nC_r + {}_nC_{r-1} \\
&= \frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r} + \frac{n(n-1)(n-2)\ldots(n-r+2)}{1.2.3\ldots(r-1)} \\
&= \frac{n(n-1)(n-2)\ldots(n-r+2)}{1.2.3\ldots r}\{(n-r+1)+r\} \\
&= \frac{(n+1)n(n-1)(n-2)\ldots(n-r+2)}{1.2.3\ldots r} \\
&= {}_{n+1}C_r.
\end{aligned}$$

240. **Greatest value of ${}_nC_r$.**

From Art. 237 we have

$${}_nC_r = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \ldots \times \frac{n-r+1}{r};$$

$$\therefore \; {}_nC_r = {}_nC_{r-1} \times \frac{n-r+1}{r}.$$

Hence ${}_nC_r \gtreqless {}_nC_{r-1}$ according as $\dfrac{n-r+1}{r} \gtreqless 1$, that is according as $$r \lesseqgtr \tfrac{1}{2}(n+1).$$

Thus the number of combinations of n things increases as r increases so long as r is less than $\frac{1}{2}(n+1)$. If $r=\frac{1}{2}(n+1)$, then ${}_nC_r={}_nC_{r-1}$, but r cannot be equal to $\frac{1}{2}(n+1)$ unless n is *odd*. If $r>\frac{1}{2}(n+1)$, the number of combinations diminishes as r increases.

Thus, if n is even, ${}_nC_r$ is greatest when $r=\frac{1}{2}n$; and if n is odd, ${}_nC_r={}_nC_{r-1}$ when $r=\frac{1}{2}(n+1)$, and these values are the greatest.

Ex. 1. Find the greatest value of ${}_8C_r$.

The greatest value is when $r=4$, and the value is

$$\frac{8.7.6.5}{1.2.3.4}=70.$$

Ex. 2. Find the greatest value of ${}_{11}C_r$.

The greatest values are when $r=6$ and $r=5$ and the values are

$$\frac{11.10.9.8.7}{1.2.3.4.5}=462.$$

EXAMPLES. LXX.

1. Find ${}_{15}P_3$. **2.** Find ${}_{10}P_4$. **3.** Find ${}_8P_8$.

4. How many different numbers can be formed by using the five figures 1, 2, 3, 4, 5?

5. How many different numbers can be formed by using one or more of the figures 1, 2, 3, 4, 5, 6?

6. In how many ways can 10 boys be put in a row so that two particular boys should not be together?

7. Of how many different things are there 720 permutations when taken all together?

8. Find the number of permutations of all the letters of each of the words *success*, *mississippi* and *algebraic*.

9. In how many ways can the nine letters *aaaabbbcc* be arranged in a row?

10. How many different numbers can be formed by using the seven figures 1, 1, 1, 2, 2, 3, 3?

11. How many permutations can be made with the letters of the word *essences*, and how many begin with *e* and end with *s*?

12. Find ${}_nP_n$, having given that ${}_nP_5 = {}_nP_4 \times 2$.

13. Find n having given that ${}_nP_6 = {}_nP_4 \times 12$.

14. Find ${}_{16}C_4$, ${}_{15}C_{12}$ and ${}_{20}C_{18}$.

15. If ${}_nC_5 = {}_nC_{10}$, find ${}_nC_2$.

16. If ${}_nC_{10} = {}_nC_{11}$, find ${}_nC_3$.

17. How many different numbers of three figures could be made by taking three of the digits 0, 1, 2, 3, 4?

18. If ${}_{2n}C_3 = {}_nC_2 \times 12$, find n.

19. If ${}_{2n}C_3 = {}_nC_4 \times 24$, find n.

20. Numbers are formed by writing the five figures 1, 2, 3, 4, 5 in every possible order. How many of these numbers are greater than 23000?

21. In a certain town there are five letter boxes. In how many ways can a person post three letters?

22. How many different sums could be made with 2 pennies, 3 shillings and 4 sovereigns?

23. How many different sums could be made with 4 pennies, 6 shillings and 5 sovereigns?

24. Shew that ${}_{n+2}C_{r+1} = {}_nC_{r+1} + 2\,{}_nC_r + {}_nC_{r-1}$.

25. Shew that in ${}_8C_4$ the number of combinations in which a particular thing occurs is equal to the number in which it does not occur.

26. Shew that in ${}_{12}C_4$ the number of combinations in which a particular thing occurs is one-third of the whole number of the combinations.

27. Shew that in ${}_{3n}C_n$ the number of the combinations in which a particular thing occurs is one-third of the whole number of the combinations.

28. There are ten candidates for 6 vacancies in a committee: in how many ways can a person vote for 6 of the candidates?

29. At an election there are five candidates and three members to be elected, and an elector may give one vote to each of not more than three candidates, in how many ways can an elector vote?

30. In how many ways can a picket of 3 men and an officer be chosen out of a company of 80 men and 3 officers.

31. In how many ways can a cricket eleven be chosen out of 30 players; and in how many different ways could two elevens be chosen to play a match with one another?

32. Out of 9 children, of whom 5 are boys and 4 girls, in how many ways can 4 be chosen, 2 being boys and 2 girls?

33. In how many ways can 2 ladies and 2 gentlemen be chosen to make a set at lawn tennis from a party of 4 ladies and 6 gentlemen?

34. In how many ways can 8 children form a ring?

35. There are n points in a plane, no three of which are in the same straight line; and the points are joined in pairs by straight lines which are produced indefinitely. How many straight lines are there, and how many triangles are formed by them?

CHAPTER XXVI.

THE BINOMIAL THEOREM.

241. We have already proved that the product of any two multinomial expressions is the sum of all the partial products obtained by multiplying any term of one expression by any term of the other.

To find the continued product of three expressions we must multiply each of the terms in the product of the first two expressions by each of the terms in the third; hence the continued product is the sum of all the partial products which can be obtained by multiplying together any term of the first, any term of the second and any term of the third.

And similarly, the continued product of any number of expressions is the sum of all the partial products which can be obtained by multiplying together any term of the first, any term of the second, any term of the third, &c.

For example, if we take a letter from each of the factors of

$$(a+b)\,(a+b)\,(a+b),$$

and multiply the three together, we shall obtain a term of the continued product; and if we do this in every possible way we shall obtain all the terms of the continued product.

We can take a every time, and we can do this in only one way, hence a^3 is a term of the continued product.

We can take a twice and b once, and we can do this in *three* ways, for the b can be taken from either of the three binomial factors; hence we have $3a^2b$.

We can take a once and b twice, and we can do this also in *three* ways; hence we have $3ab^2$.

Finally, we can take b every time, and this can be done in only one way; hence we have b^3.

Thus the continued product is

$$a^3 + 3a^2b + 3ab^2 + b^3,$$

so that $$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

242. **Binomial Theorem.** Suppose we have n factors each of which is $a + b$. If we take a letter from each of the factors of

$$(a+b)(a+b)(a+b)\ldots\ldots$$

and multiply them all together, we shall obtain a term of the continued product; and if we do this in every possible way we shall obtain all the terms of the continued product.

Now we can take the letter a every time, and this can be done in only one way: hence a^n is a term of the product.

The letter b can be taken once, and a the remaining $(n-1)$ times, and the number of ways in which one b can be taken is the number of ways of taking 1 out of n things, so that the number is ${}_nC_1$: hence we have

$${}_nC_1 \,.\, a^{n-1}b.$$

Again, the letter b can be taken twice, and a the remaining $(n-2)$ times, and the number of ways in which two b's can be taken is the number of ways of taking 2 out of n things, so that the number is ${}_nC_2$: hence we have

$${}_nC_2 \,.\, a^{n-2}b^3.$$

And, in general, b can be taken r times (where r is any positive integer not greater than n) and a the remaining $(n-r)$ times, and the number of ways in which r b's can be taken is the number of ways of taking r out of n things, so that the number is ${}_nC_r$: hence we have

$${}_nC_r \,.\, a^{n-r} b^r.$$

The letter b can be taken every time in only one way; hence we have the term b^n, which agrees with the result obtained by putting $r = n$ in ${}_nC_r \,.\, a^{n-r} b^r$, since ${}_nC_n = 1$.

Thus $\quad (a+b)(a+b)(a+b)\dots$to n factors

$$= a^n + {}_nC_1 \,.\, a^{n-1} b + {}_nC_2\, a^{n-2} b^2 + \dots + {}_nC_r\, a^{n-r} b^r + \dots + b^n.$$

Hence, when n is any positive integer, we have

$$(a+b)^n = a^n + {}_nC_1 \,.\, a^{n-1} b + {}_nC_2 \,.\, a^{n-2} b^2 + \dots \\ + {}_nC_r\, a^{n-r} b^r + \dots + b^n.$$

The above formula is called the *Binomial Theorem.*

The series on the right is called the *expansion* of

$$(a+b)^n.$$

If we substitute the known values [Art. 237] of ${}_nC_1$, ${}_nC_2$, &c. in the series on the right we obtain the form in which the theorem is usually written, namely

$$(a+b)^n = a^n + na^{n-1} b + \frac{n(n-1)}{1 \,.\, 2} a^{n-2} b^2 + \dots \\ + \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} a^{n-r} b^r + \dots + b^n.$$

243. **General term.** Any term of the expansion of $(a+b)^n$ will be found by giving a suitable value to r in

$$\frac{n(n-1)(n-2)\dots(n-r+1)}{\underline{|r}} a^{n-r} b^r.$$

On this account, the above is called the *general term* of the expansion. It should be noticed that the term is the $(r+1)$th from the beginning.

244. If we put $n=2$ in the formula for the expansion of a binomial, we have

$$(a+b)^2=a^2+2ab+b^2.$$

If we put $n=3$, we have

$$(a+b)^3=a^3+\frac{3}{1}a^2b+\frac{3 \,.\, 2}{1 \,.\, 2}ab^2+b^3$$
$$=a^3+3a^2b+3ab^2+b^3.$$

If we put $n=4$, we have

$$(a+b)^4=a^4+\frac{4}{1}a^3b+\frac{4 \,.\, 3}{1 \,.\, 2}a^2b^2+\frac{4 \,.\, 3 \,.\, 2}{1 \,.\, 2 \,.\, 3}ab^3+b^4$$
$$=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

If we put $2x$ for a, and $-3y$ for b, and $n=5$, we have

$$(2x-3y)^5=(2x)^5+5\,(2x)^4\,(-3y)+\frac{5 \,.\, 4}{1 \,.\, 2}(2x)^3\,(-3y)^2$$
$$+\frac{5 \,.\, 4 \,.\, 3}{1 \,.\, 2 \,.\, 3}(2x)^2\,(-3y)^3+\frac{5 \,.\, 4 \,.\, 3 \,.\, 2}{1 \,.\, 2 \,.\, 3 \,.\, 4}(2x)\,(-3y)^4+(-3y)^5$$
$$=32x^5-240x^4y+720x^3y^2-1080x^2y^3+810xy^4-243y^5.$$

245. **Proof by Induction.** The Binomial Theorem may also be proved in the following manner.

We have to prove that, when n is any positive integer,

$$(a+b)^n=a^n+na^{n-1}b+\frac{n\,(n-1)}{1 \,.\, 2}a^{n-2}b^2+\dots$$
$$+\frac{\underline{|n}}{\underline{|r}\;\underline{|n-r}}a^{n-r}b^r+\dots+b^n,$$

or that

$$(a+b)^n=a^n+{}_nC_1\,a^{n-1}b+{}_nC_2\,a^{n-2}b^2+\dots$$
$$+{}_nC_{r-1}\,a^{n-r+1}b^{r-1}+{}_nC_r\,a^{n-r}b^r+\dots+b^n.$$

Now if we *assume* that the theorem is true when the index is n, and multiply by another factor $a+b$, we shall have, when the terms in the product which contain the same powers of a and b are collected,

$$(a+b)^{n+1}=a^{n+1}+(1+{}_nC_1)\,a^nb+({}_nC_1+{}_nC_2)\,a^{n-1}b^2+\dots$$
$$+({}_nC_{r-1}+{}_nC_r)\,a^{n-r+1}b^r+\dots+b^{n+1}.$$

Now $$1 + {}_nC_1 = 1 + n = {}_{n+1}C_1,$$

$$ {}_nC_1 + {}_nC_2 = n + \frac{n(n-1)}{2} = \frac{(n+1)n}{2} = {}_{n+1}C_2,$$

and, for any value of r,

$$ {}_nC_{r-1} + {}_nC_r = {}_{n+1}C_r{}^*. \qquad \text{[Art. 239.]}$$

Hence $$(a+b)^{n+1} = a^{n+1} + {}_{n+1}C_1 \,.\, a^n b + {}_{n+1}C_2 \,.\, a^{n-1} b^2 + \dots + {}_{n+1}C_r \,.\, a^{n+1-r} b^r + \dots + b^{n+1}.$$

Thus, *if* the theorem be true for any value of n, it will be true for the next greater value.

Now the theorem is obviously true when $n = 1$. Hence it must be true when $n = 2$; and being true when $n = 2$, it must be true when $n = 3$; and so on indefinitely. The theorem is therefore true for *all* positive integral values of n.

The above method of proof is frequently employed in Algebra, and is called the method of *Induction.*

246. In the expansion of $(a+b)^n$ by the binomial theorem, the $(r+1)$th term from the beginning and the $(r+1)$th term from the end are respectively

$$ {}_nC_r \, a^{n-r} b^r \text{ and } {}_nC_{n-r} \, a^r b^{n-r}.$$

But ${}_nC_r = {}_nC_{n-r}$, for all values of r. [Art. 238.]

Hence in the expansion of $(a+b)^n$, *the coefficients of any two terms equidistant from the beginning and the end are the same, so that the coefficients in order are the same when read backwards as when read forwards.*

247. If in the formula of Art. 242 we put $a = 1$ and $b = x$ we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \,.\, 2} x^2 + \dots + \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} x^r + \dots + x^n.$$

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* In writing out a proof of the Binomial Theorem it would be well to insert Art. 239 at this stage. This is the fundamental point in the proof of the Binomial Theorem, but it is frequently omitted altogether.

This is the most convenient form of the Binomial Theorem, and the one which is generally employed.

It should be noticed that the above form of the Binomial Theorem includes all cases; if, for example, we want to find $(a+b)^n$, we have

$$(a+b)^n = \left\{a\left(1+\frac{b}{a}\right)\right\}^n = a^n\left(1+\frac{b}{a}\right)^n = a^n\left(1+n\,\frac{b}{a}+\&\text{c.}\right)$$
$$= a^n + na^{n-1}b + \&\text{c.}$$

248. **Greatest term.** In the expansion of $(1+x)^n$ by the binomial theorem, the $(r+1)$th term is formed from the rth by multiplying by $\dfrac{n-r+1}{r}x$.

Now $\dfrac{n-r+1}{r}x = \left(\dfrac{n+1}{r}-1\right)x$, and $\dfrac{n+1}{r}$ clearly diminishes as r increases; hence $\dfrac{n-r+1}{r}x$ diminishes as r is increased. If $\dfrac{n-r+1}{r}x$ be less than 1 for any value of r, the $(r+1)$th term will be less than the rth. In order therefore that the rth term of the expansion may be the *greatest* we must have

$$\frac{n-r+1}{r}x < 1 \text{ and } \frac{n-\overline{r-1}+1}{r-1}x > 1.$$

Hence $$r > \frac{(n+1)x}{x+1}, \text{ and } r < \frac{(n+1)x}{x+1}+1.$$

If $r = \dfrac{(n+1)x}{x+1}$, then $\dfrac{n-r+1}{r}x = 1$, and there is no *one* term which is the greatest, but the rth and $(r+1)$th terms are equal and are greater than any other terms.

The absolute values of the terms in the expansion of

$(1+x)^n$ will not be altered by changing the sign of x; and hence the rth term of $(1-x)^n$ will be greatest when the rth term of $(1+x)^n$ is greatest.

Ex. 1. Find the greatest term in the expansion of $(1+x)^{10}$, when $x=\frac{1}{3}$.

The rth term is the greatest, if $r>\frac{11}{4}$ and $r<\frac{11}{4}+1$. Hence the third term is the greatest.

Ex. 2. Find the greatest term in the expansion of $(2+3x)^{12}$, when $x=\frac{1}{2}$.

$$(2+3x)^{12}=\{2\,(1+\tfrac{3}{2}x)\}^{12}=2^{12}\,(1+\tfrac{3}{2}x)^{12}.$$

Hence the r term is the greatest if $r>\frac{39}{7}$ and $r<\frac{39}{7}+1$. Hence the sixth term is the greatest.

The greatest *coefficient* of a binomial expansion has already been found in Art. 240.

EXAMPLES. LXXI.

Write out the following expansions:

1. $(x+a)^6$. **2.** $(1-x^2)^5$. **3.** $(3x-2y)^4$.

4. $(2a+3a^2)^4$. **5.** $(2x^2-1)^6$. **6.** $(y-x)^7$.

7. Find the third term of $(a-3b)^{10}$.

8. Find the fifth term of $(2x-x^2)^{12}$.

9. Find the sixth term of $(2a-\frac{1}{2})^8$.

10. Find the seventh term of $(1-x)^{10}$.

11. Find the eighteenth term of $(1+x)^{20}$.

12. Find the twenty-first term of $(1-x)^{22}$.

13. Expand $(a+\sqrt{b})^4+(a-\sqrt{b})^4$.

14. Expand $(a+\sqrt{b})^6+(a-\sqrt{b})^6$.

15. Expand $(a+\sqrt{b})^8+(a-\sqrt{b})^8$.

16. Find the middle term of $(1+x)^8$.

17. Find the middle term of $(1+x)^{10}$.

18. Find the middle term of $(2x-3y)^8$.

19. Shew that, in the expansion of $(1+x)^{m+n}$, the coefficients of x^m and x^n are equal.

20. Expand $\left(x^2+\frac{1}{x^2}\right)^8$.

21. Find the 11th term in the expansion of $\left(4x-\frac{1}{2\sqrt{x}}\right)^{15}$.

22. Find the 17th term in the expansion of $\left(x^2-\frac{1}{x}\right)^{20}$.

23. Find the coefficient of x^n in the expansion of $\left(x^2-\frac{a^3}{x}\right)^{2n}$.

24. Write down the two middle terms in the expansion of

$$\left(x+\frac{1}{x}\right)^{2n+1}.$$

25. Find the term in the expansion of $(a+x)^8$ which has the greatest coefficient.

26. Find the terms in the expansion of $(a+x)^9$ which have the greatest coefficients.

27. Find the greatest term in the expansion of $(1+2x)^{20}$ when $x=\frac{1}{3}$.

28. Find the greatest term in the expansion of $(1+3x)^{18}$ when $x=\frac{2}{5}$.

29. Find the greatest term in the expansion of $(3+4x)^{12}$ when $x=\frac{1}{2}$.

30. There are two consecutive terms in the expansion of $(5x+7)^{23}$ which have the same coefficient: which are they?

31. Write down the coefficient of x^r in the expansion of $(ax-b)^n$.

32. Shew that the coefficient of x^n in the expansion of $(1+x)^{2n}$ is double the coefficient of x^n in the expansion of $(1-x)^{2n-1}$.

33. Shew that the middle term of $(1+x)^{2n}$ is $\frac{1.3.5\ldots(2n-1)}{\underline{|n}}2^n x^n$.

34. Employ the binomial theorem to find 99^4 and 999^3.

249. **Properties of the Coefficients.** We now proceed to consider some properties of the coefficients of a binomial expansion.

It is often convenient to write the binomial theorem in the following form:

$$(1+x)^n = c_0 + c_1x + c_2x^2 + \ldots + c_rx^r + \ldots + c_nx^n,$$

where

$$c_0 = c_n = 1, \;\; c_1 = c_{n-1} = n, \text{ and } c_r = c_{n-r} = \frac{|\underline{n}}{|\underline{r}\,|\underline{n-r}}.$$

I. Put $x = 1$ in the above formula; and we have

$$2^n = c_0 + c_1 + c_2 + \dots + c_r + \dots c_n.$$

Thus *the sum of the coefficients in the expansion of* $(1+x)^n$ *is* 2^n.

II. Again, put $x = -1$; and we have

$$(1-1)^n = c_0 - c_1 + c_2 - c_3 + \dots;$$

$$\therefore \;\; 0 = (c_0 + c_2 + c_4 + \dots) - (c_1 + c_3 + c_5 + \dots).$$

Thus *the sum of the coefficients of the odd terms of a binomial expansion is equal to the sum of the coefficients of the even terms.*

III. Since $c_r = c_{n-r}$, [Art. 246] we may write the binomial theorem in either of the following ways:

$$(1+x)^n = c_0 + c_1x + c_2x^2 + \dots + c_rx^r + \dots c_nx^n,$$

or $(1+x)^n = c_n + c_{n-1}x + c_{n-2}x^2 + \dots c_{n-r}x^r + \dots + c_1x^{n-1} + c_0x^n.$

The coefficient of x^n in the product of the two series on the right is equal to

$$c_0^2 + c_1^2 + c_2^2 + \dots + c_n^2.$$

Hence* $c_0^2 + c_1^2 + \dots + c_r^2 + \dots c_n^2$ is equal to the coefficient of x^n in $(1+x)^n \times (1+x)^n$, that is in $(1+x)^{2n}$, and this coefficient is $\dfrac{|\underline{2n}}{|\underline{n}\,|\underline{n}}$.

Hence *the sum of the squares* of the coefficients in the expansion of $(1+x)^n$ is equal to $\dfrac{|\underline{2n}}{|\underline{n}\,|\underline{n}}$.

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* See Art. 272.

Ex. 1. Shew that

$$c_1 + 2c_2 + 3c_3 + \ldots + rc_r + \ldots + nc_n = n2^{n-1}.$$

$$\begin{aligned}
&c_1 + 2c_2 + 3c_3 + \ldots + rc_r + \ldots + nc_n \\
&= n + 2 \cdot \frac{n(n-1)}{1 \cdot 2} + 3 \cdot \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \ldots + r \frac{\underline{|n}}{\underline{|r}\ \underline{|n-r}} + \ldots + n \\
&= n\left\{1 + (n-1) + \frac{(n-1)(n-2)}{1 \cdot 2} + \ldots + \frac{\underline{|n-1}}{\underline{|r-1}\ \underline{|n-r}} + \ldots + 1\right\} \\
&= n(1+1)^{n-1} \\
&= n2^{n-1}.
\end{aligned}$$

Ex. 2. Shew that

$$c_0 - \frac{1}{2}c_1 + \frac{1}{3}c_2 - \ldots + (-1)^n \frac{c_n}{n+1} = \frac{1}{n+1}.$$

$$\begin{aligned}
&c_0 - \frac{1}{2}c_1 + \frac{1}{3}c_2 - \frac{1}{4}c_3 - \ldots + (-1)^n \frac{c_n}{n+1} \\
&= 1 - \frac{1}{2} \cdot n + \frac{1}{3}\frac{n(n-1)}{1 \cdot 2} - \frac{1}{4}\frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} + \ldots + (-1)^n \frac{1}{n+1} \\
&= \frac{1}{n+1}\left\{(n+1) - \frac{(n+1)n}{1 \cdot 2} + \frac{(n+1)n(n-1)}{1 \cdot 2 \cdot 3} - \ldots + (-1)^n\right\} \\
&= \frac{1}{n+1} - \frac{1}{n+1}\left\{1 - (n+1) + \frac{(n+1)n}{1 \cdot 2} - \ldots + (-1)^{n+1}\right\} \\
&= \frac{1}{n+1} - \frac{1}{n+1}(1-1)^{n+1} \\
&= \frac{1}{n+1}.
\end{aligned}$$

250. *To find the continued product of* n *binomial factors of the form* $x + a$, $x + b$, $x + c$, &c.

It will be convenient to use the following notation:

S_1 is written for $a + b + c + \ldots$, the sum of all the letters taken *one* at a time. S_2 is written for $ab + ac + bc + \ldots$, the sum of all the products which can be obtained by taking the letters *two* at a time. And, in general, S_r is written for the sum of all the products which can be obtained by taking the letters r at a time.

Now if we take a letter from each of the binomial factors of

$$(x+a)(x+b)(x+c)(x+d)\ldots,$$

and multiply them all together, we shall obtain a term of the continued product; and, if we do this in every possible way, we shall obtain all the terms of the continued product.

We can take x every time, and this can be done in only one way; hence x^n is a term of the continued product.

We can take any *one* of the letters a, b, c,..., and x from all the remaining $n-1$ binomial factors; we thus have the terms ax^{n-1}, bx^{n-1}, cx^{n-1}, and on the whole $S_1 . x^{n-1}$.

Again, we can take any *two* of the letters a, b, c,..., and x from all the remaining $n-2$ binomial factors; we thus have the terms abx^{n-2}, acx^{n-2}, etc., and on the whole $S_2 . x^{n-2}$.

And, in general, we can take any r of the letters a, b, c,..., and x from all the remaining $n-r$ binomial factors; and we thus have $S_r . x^{n-r}$.

Hence $$(x+a)(x+b)(x+c)(x+d)\ldots$$
$$= x^n + S_1 . x^{n-1} + S_2 . x^{n-2} + \ldots + S_r . x^{n-r} + \ldots,$$

the last term being $abcd\ldots$, the product of all the letters a, b, c, d, etc.

By changing the signs of a, b, c, etc. the signs of S_1, S_3, S_5, etc. will be changed, but the signs of S_2, S_4, S_6, etc. will be unaltered. Hence we have

$$(x-a)(x-b)(x-c)(x-d)\ldots$$
$$= x^n - S_1 . x^{n-1} + S_2 . x^{n-2} - S_3 . x^{n-3} + \ldots$$
$$+ (-1)^r S_r . x^{n-r} + \ldots + (-1)^n abcd\ldots.$$

251. The Multinomial Theorem. The expansion of $(a+b+c+\dots)^n$ can be found by the method of Art. 242.

We know that the continued product

$$(a+b+c+\dots)(a+b+c+\dots)(a+b+c+\dots)\dots$$

is the sum of all the partial products which can be obtained by multiplying any term from the first multinomial factor, any term from the second, any term from the third, etc. If there are n of the multinomial factors every term of the required expansion must be of the nth degree, and therefore all the terms must be of the form $a^r b^s c^t\dots$, where each of r, s, t, etc. is zero or a positive integer, and $r+s+t+\dots=n$.

Now the term a^r, b^s, $c^t\dots$ will be obtained by taking a from any r of the n multinomial factors, which can be done in ${}_nC_r$ different ways; then taking b from any s of the remaining $n-r$ factors, which can be done in ${}_{n-r}C_s$ different ways; then taking c from any t of the remaining $n-r-s$ factors, which can be done in ${}_{n-r-s}C_t$ different ways; and so on. Hence the total number of ways in which the term $a^r b^s c^t\dots$ will be obtained, that is the coefficient of the term in the required expansion, must be

$${}_nC_r \times {}_{n-r}C_s \times {}_{n-r-s}C_t \times \dots,$$

that is

$$\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} \times \frac{\underline{|n-r}}{\underline{|s}\,\underline{|n-r-s}} \times \frac{\underline{|n-r-s}}{\underline{|t}\,\underline{|n-r-s-t}} \times \dots = \frac{\underline{|n}}{\underline{|r}\,\underline{|s}\,\underline{|t}\dots}$$

Hence the general term in the expansion of

$$(a+b+c+\dots)^n \text{ is}$$

$$\frac{\underline{|n}}{\underline{|r}\,\underline{|s}\,\underline{|t}\dots} a^r b^s c^t,$$

where each of r, s, t, etc. is zero or a positive integer and $r+s+t+\dots=n$.

Ex. 1. Find the coefficient of abc in $(a+b+c)^3$.

Here $n=3$, $r=s=t=1$. Hence the required coefficient is $\frac{\underline{|3}}{\underline{|1}\,\underline{|1}\,\underline{|1}}=6$.

Ex. 2. Find the coefficients of a^4b, a^3b^2 and a^2b^2c in $(a+b+c)^5$.

We have the terms

$$\frac{\underline{|5}}{\underline{|4}\,\underline{|1}}a^4b,\quad \frac{\underline{|5}}{\underline{|3}\,\underline{|2}}a^3b^2 \text{ and } \frac{\underline{|5}}{\underline{|2}\,\underline{|2}\,\underline{|1}}a^2b^2c.$$

Hence the required coefficients are respectively 5, 10 and 30.

EXAMPLES. LXXII.

In the following examples c_0, c_1, c_2, are the coefficients of x^0, x^1, x^2... in the expansion of $(1+x)^n$.

1. Prove that $c_1-2c_2+3c_3-\dots+(-1)^{n-1}nc_n=0$.

2. Prove that $c_0-2c_1+3c_2-\dots+(-1)^n(n+1)c_n=0$.

3. Prove that $c_0+\frac{1}{2}c_1+\frac{1}{3}c_2+\dots+\frac{1}{n+1}c_n=\frac{2^{n+1}-1}{n+1}$.

4. Prove that $\frac{c_1}{c_0}+2\frac{c_2}{c_1}+3\frac{c_3}{c_2}+\dots+n\frac{c_n}{c_{n-1}}=\frac{1}{2}n(n+1)$.

5. Prove that $\frac{c_0}{1}+\frac{c_2}{3}+\frac{c_4}{5}+\frac{c_6}{7}+\dots=\frac{2^n}{n+1}$.

6. Prove that

$$c_0+c_1x+2c_2x^2+\dots+rc_rx^r\dots+nc_nx^n=1+nx(1+x)^{n-1}.$$

7. Prove that $c_0c_1+c_1c_2+c_2c_3+\dots+c_{n-1}c_n=\frac{\underline{|2n}}{\underline{|n+1}\,\underline{|n-1}}$.

8. Prove that $c_0c_2+c_1c_3+c_2c_4+\dots+c_{n-2}c_n=\frac{\underline{|2n}}{\underline{|n+2}\,\underline{|n-2}}$.

9. Prove that $c_0^2-c_1^2+c_2^2-\dots+(-1)^nc_n^2$ is equal to 0 if n be odd, and to $\frac{\underline{|n}}{\underline{|\tfrac{1}{2}n}\,\underline{|\tfrac{1}{2}n}}$ if n be even.

10. Prove that

$$c_0^2+2c_1^2+3c_2^2+\ldots+(n+1)\,c_n^2=\frac{(n+2)\,\underline{|2n-1}}{\underline{|n-1}\;\underline{|n}}.$$

11. Prove that

$$c_1^2+2c_2^2+3c_3^2+\ldots+nc_n^2=\frac{\underline{|2n-1}}{\underline{|n-1}\;\underline{|n-1}}.$$

12. Prove that

$$c_1-\frac{1}{2}c_2+\frac{1}{3}c_3-\ldots+(-1)^{n-1}\frac{1}{n}c_n=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}.$$

13. Prove that

$$\frac{1}{x}-\frac{c_1}{x+1}+\frac{c_2}{x+2}-\ldots+(-1)^n\frac{c_n}{x+n}=\frac{\underline{|n}}{x\,(x+1)\,(x+2)\ldots(x+n)}.$$

14. Shew that, if there be a middle term in the expansion of $(1+x)^n$, its coefficient will be even.

THE BINOMIAL THEOREM. ANY INDEX.

252. It was proved in Art. 247 that, when n is any positive integer,

$$(1+x)^n=1+nx+\frac{n\,(n-1)}{1\,.\,2}x^2+\ldots$$
$$+\frac{n\,(n-1)\ldots(n-r+1)}{\underline{|r}}x^r+\ldots$$

The above result holds good for *all values* of n, provided only that x is less than 1.

When n is a positive integer the series on the right stops at the $(n+1)$th term; but, when n is not a positive integer, the series is endless, for no one of the factors n, $n-1$, $n-2$, etc. can in that case be zero.

253. Before proceeding to give a proof of the Binomial Theorem when the index is not a positive integer, we give some examples of its application.

Ex. 1. Expand $(1+x)^{-1}$ by the binomial theorem. Put $n=-1$ in the formula, and we have

$$(1+x)^{-1}=1+(-1)\,x+\frac{(-1)(-2)}{1\,.\,2}\,x^2+\frac{(-1)(-2)(-3)}{1\,.\,2\,.\,3}\,x^3+\ldots$$
$$+\frac{(-1)(-2)\ldots(-r)}{\underline{|r}}\,x^r+\ldots$$
$$=1-x+x^2-x^3+\ldots+(-1)^r x^r+\ldots.$$

Ex. 2. Expand $(1+x)^{-2}$.

We have

$$(1+x)^{-2}=1+(-2)\,x+\frac{(-2)(-3)}{1\,.\,2}\,x^2+\frac{(-2)(-3)(-4)}{1\,.\,2\,.\,3}\,x^3+\ldots$$
$$+\frac{(-2)(-3)(-4)\ldots(-r-1)}{\underline{|r}}\,x^r+\ldots$$
$$=1-2x+3x^2-4x^3+\ldots+(-1)^r(r+1)\,x^r+\ldots.$$

Ex. 3. Expand $(1+x)^{-3}$.

$$(1+x)^{-3}=1+(-3)\,x+\frac{(-3)(-4)}{1\,.\,2}\,x^2+\frac{(-3)(-4)(-5)}{1\,.\,2\,.\,3}\,x^3+\ldots$$
$$+\frac{(-3)(-4)(-5)\ldots(-r-2)}{\underline{|r}}\,x^r+\ldots$$
$$=\frac{1}{1\,.\,2}\{1\,.\,2-2\,.\,3\,x+3\,.\,4\,x^2-4\,.\,5\,x^3+\ldots$$
$$+(-1)^r(r+1)(r+2)\,x^r+\ldots\}.$$

Ex. 4. Expand $(1-x)^{-\frac{1}{2}}$.

$$(1-x)^{-\frac{1}{2}}=1+(-\tfrac{1}{2})(-x)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{1\,.\,2}(-x)^2+\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1\,.\,2\,.\,3}(-x)^3$$
$$+\ldots+\frac{(-\frac{1}{2})(-\frac{3}{2})\ldots(-\frac{1}{2}-r+1)}{\underline{|r}}(-x)^r+\ldots$$
$$=1+\frac{1}{2}x+\frac{1\,.\,3}{2\,.\,4}x^2+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}x^3+\ldots$$
$$+\frac{1\,.\,3\,.\,5\ldots(2r-1)}{2\,.\,4\,.\,6\ldots2r}x^r+\ldots.$$

254. **Euler's proof of the Binomial Theorem.** Represent for shortness any series of the same form as the binomial series, that is any series of the form

$$1+nx+\frac{n(n-1)}{1\,.\,2}x^2+\ldots,\ \text{by}\ f(n).$$

Thus $$f(n) = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \ldots\ldots\ldots\ldots\ldots \text{(i)},$$

$$f(m) = 1 + mx + \frac{m(m-1)}{1 \cdot 2}x^2 + \ldots\ldots\ldots\ldots \text{(ii)},$$

and

$$f(m+n) = 1 + (m+n)x + \frac{(m+n)(m+n-1)}{1 \cdot 2}x^2 + \ \ldots \text{(iii)}.$$

Now, if the series on the right of (i) and (ii) be multiplied, and the product be arranged according to ascending powers of x, the result must involve m and n in the same manner whatever their values may be. But, when m and n are positive integers, we know that $f(m)$ is $(1+x)^m$ and that $f(n)$ is $(1+x)^n$, and the product of $f(m)$ and $f(n)$ is therefore in this case $(1+x)^{m+n}$, which again, as $m+n$ is a positive integer, is $f(m+n)$. Hence *when* m *and* n *are positive integers* the product $f(m) \times f(n)$ is $f(m+n)$; and as the *form* of the product is the same for *all values* of m and n, it follows* that

$$f(m) \times f(n) = f(m+n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\alpha),$$

for *all values* of m and n.

By continued application of (α), we have

$$\begin{aligned} f(m) \times f(n) \times f(p) \times \ldots &= f(m+n) \times f(p) \times \ldots \\ &= f(m+n+p+\ldots). \end{aligned}$$

Now let $m = n = p = \ldots = \frac{r}{s}$, where r and s are positive integers; then, taking s factors, we have

$$f\left(\frac{r}{s}\right) \times f\left(\frac{r}{s}\right) \times \ldots \text{ to } s \text{ factors} = f\left(\frac{r}{s} + \frac{r}{s} + \ldots \text{ to } s \text{ terms}\right);$$

$$\therefore \left\{f\left(\frac{r}{s}\right)\right\}^s = f(r).$$

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* Provided, however, that $f(m)$ and $f(n)$ are absolutely convergent series. [See *Treatise on Algebra*, Art. 280.]

But since r is a positive integer, $f(r) = (1+x)^r$.

$$\therefore\ (1+x)^r = \left\{f\left(\frac{r}{s}\right)\right\}^s;$$

$$\therefore\ (1+x)^{\frac{r}{s}} = f\left(\frac{r}{s}\right).$$

This proves the Binomial Theorem for any positive fractional exponent.

Next, assuming that the binomial theorem is true for any positive index, it can be proved to be true also for any negative index.

For, from (α),

$$f(-n) \times f(n) = f(-n+n) = f(0).$$

Hence, as $f(0)$ is clearly 1, we have

$$f(-n) = \frac{1}{f(n)} = \frac{1}{(1+x)^n}, \text{ since } n \text{ is positive,}$$
$$= (1+x)^{-n}.$$

Thus $(1+x)^{-n} = f(-n)$, which proves the theorem for any negative index.

Ex. 1. Shew that the coefficients of x^{n-1} in the expansions of $(1-x)^{-n}$ and $(1+x)^{2n-2}$ are equal, n being any positive integer.

The coefficients are

$$\frac{(-n)(-n-1)\ldots(-n-n+2)}{\underline{|n-1}}(-x)^{n-1} \text{ and } \frac{\underline{|2n-2}}{\underline{|n-1}\,\underline{|n-1}}x^{n-1}.$$

$$\text{The former} = \frac{n(n+1)\ldots(2n-2)}{\underline{|n-1}}x^{n-1} = \frac{\underline{|2n-2}}{\underline{|n-1}\,\underline{|n-1}}x^{n-1}.$$

Ex. 2. Find $\sqrt{101}$ by the binomial theorem.

$$\sqrt{101} = \sqrt{\{100(1+\tfrac{1}{100})\}} = 10(1+\tfrac{1}{100})^{\frac{1}{2}}$$

$$= 10\left\{1 + \frac{1}{2} \times \cdot 01 + \frac{\frac{1}{2}(-\frac{1}{2})}{1\,.\,2} \times \cdot 0001 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\,.\,2\,.\,3} \times \cdot 000001 + \ldots\right\}$$

$$= 10\{1 + \cdot 005 - \cdot 0000125 + \cdot 0000000625\} = 10 \cdot 04987562 \ldots.$$

Ex. 2. Find the coefficient of x^3 in the expansion of

$$(1-2x+4x^2)^{-2}$$

according to ascending powers of x.

$$\begin{aligned}(1-2x+4x^2)^{-2} &= \{1-2x(1-2x)\}^{-2}\\ &= 1+(-2)\{-2x(1-2x)\}+\frac{(-2)(-3)}{1\,.\,2}\{-2x(1-2x)\}^2\\ &\quad+\frac{(-2)(-3)(-4)}{1\,.\,2\,.\,3}\{-2x(1-2x)\}^3+\ldots\\ &= 1+4x(1-2x)+12x^2(1-2x)^2+32x^3(1-2x)^3+\ldots\\ &= 1+4x+4x^2-16x^3+\ldots.\end{aligned}$$

Ex. 3. Shew that, if $(1+x+x^2+x^3)^{-2}$ be expanded in a series of ascending powers of x, the coefficients of x^3 and x^7 will be zero.

$$\begin{aligned}(1+x+x^2+x^3)^{-2} &= \left(\frac{1-x^4}{1-x}\right)^{-2}=(1-x)^2(1-x^4)^{-2}\\ &= (1-2x+x^2)(1+2x^4+3x^8+4x^{12}+\ldots).\end{aligned}$$

255. **The Exponential Theorem.** The Exponential Theorem is the theorem by means of which a^x can be expanded in a series of ascending powers of x.

Let $f(m)$ denote the series $1+\frac{m}{\underline{|1}}+\frac{m^2}{\underline{|2}}+\ldots+\frac{m^r}{\underline{|r}}+\ldots$.

Then $f(1)=1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\ldots$

The important series on the right is always denoted by e. It is easily seen that the value of e lies between 2 and 3, for e is obviously greater than 2 and is less than $1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots$, that is less than 3. The actual value of e can be found to be $2{\cdot}71828\ldots$.

Consider the three series:

$$f(m)=1+m+\frac{m^2}{\underline{|2}}+\ldots+\frac{m^r}{\underline{|r}}+\ldots,$$

$$f(n)=1+n+\frac{n^2}{\underline{|2}}+\ldots+\frac{n^s}{\underline{|s}}+\ldots,$$

and

$$f(m+n)=1+(m+n)+\frac{(m+n)^2}{\underline{|2}}+\ldots+\frac{(m+n)^p}{\underline{|p}}+\ldots.$$

The coefficient of $m^r n^s$ in $f(m)\times f(n)$ is clearly $\dfrac{1}{\underline{|r}\,\underline{|s}}$ and in $f(m+n)$ the term $m^r n^s$ can only occur in $\dfrac{(m+n)^{r+s}}{\underline{|r+s}}$, and by the binomial theorem for a positive integral index, its coefficient will be $\dfrac{1}{\underline{|r+s}}\cdot\dfrac{\underline{|r+s}}{\underline{|r}\,\underline{|s}}$, that is $\dfrac{1}{\underline{|r}\,\underline{|s}}$.

Hence, whatever m and n may be, the coefficients of $m^r n^s$ in $f(m)\times f(n)$ and in $f(m+n)$ are the same for all values of r and s; it therefore follows that

$$f(m)\times f(n)=f(m+n)\ldots\ldots\ldots(\alpha),$$

for all values of m and n.

By continued application of (α), we have

$$\begin{aligned} f(m)\times f(n)\times f(p)\times\ldots &= f(m+n)\times f(p)\times\ldots \\ &= f(m+n+p+\ldots). \end{aligned}$$

Hence, if x be any *positive integer*,

$$\begin{aligned} f(1)\times f(1)\times f(1)\times\ldots\text{ to } x \text{ factors} & \\ &= f(1+1+1+\ldots\text{ to } x\text{ terms}); \end{aligned}$$

$$\therefore\ \{f(1)\}^x=f(x)\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

that is $e^x=f(x)$.

Next let x be a positive fraction $\dfrac{p}{q}$, where p and q are positive integers; then

$$\begin{aligned} f\left(\frac{p}{q}\right)\times f\left(\frac{p}{q}\right)\times f\left(\frac{p}{q}\right)+\ldots\text{ to } q \text{ factors} & \\ &= f\left(\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\ldots\text{ to } q\text{ terms}\right); \end{aligned}$$

$$\therefore \left\{f\left(\frac{p}{q}\right)\right\}^q = f(p)$$

$$= e^p, \text{ since } p \text{ is a positive integer;}$$

$$\therefore f\left(\frac{p}{q}\right) = e^{\frac{p}{q}}.$$

Hence $e^x = f(x)$ for *all positive values* of x.

Lastly, let x be negative and equal to $-y$.

Then, from (α), $f(-y) \times f(y) = f(0)$,
and $f(0)$ is clearly equal to 1.

Hence $f(-y) = \dfrac{1}{f(y)} = \dfrac{1}{e^y}$, since y is positive

$$= e^{-y}.$$

Hence, whatever x may be,

$$e^x = f(x) = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots.$$

Now let $a = e^\lambda$; then $a^x = e^{\lambda x}$;

$$\therefore a^x = 1 + x\lambda + \frac{x^2\lambda^2}{\underline{|2}} + \frac{x^3\lambda^3}{\underline{|3}} + \ldots,$$

where λ is such that $a = e^\lambda$.

EXAMPLES. LXXIII.

1. Expand $(1-x)^{-4}$ to 5 terms.
2. Expand $(1+2x)^{-4}$ to 5 terms.
3. Expand $(2-x)^{-3}$ to 6 terms.
4. Expand $(1-3x)^{-\frac{2}{3}}$ to 5 terms.
5. Expand $(1-5x)^{\frac{3}{5}}$ to 5 terms.

Find the general term in the expansion of each of the following:

6. $(1-x)^{-5}$, 7. $(1-x)^{-n}$, 8. $(1-x)^{-\frac{3}{2}}$, 9. $(1+x)^{\frac{3}{2}}$,
10. $(1+x)^{-\frac{2}{3}}$, 11. $(1-2x)^{-\frac{5}{2}}$, 12. $(1+3x)^{-\frac{4}{3}}$.

13. Find the coefficient of x^6 in the expansion of $\dfrac{a}{\sqrt{(a^2 - x^2)}}$ according to ascending powers of x.

14. Expand $(1-3x)^{\frac{2}{3}}$ to 5 terms, and write down the general term in its simplest form.

15. Find the coefficients of x^3 and x^4 in the expansion of

$$(1-2x+3x^2)^{-1}.$$

16. Find the coefficients of x^3 and x^4 in the expansion of

$$(1-x+3x^2)^{-\frac{1}{2}}.$$

17. Find the first negative term of $(1+3x)^{\frac{7}{2}}$.

18. Find the first negative term of $(1+3x)^{\frac{19}{6}}$.

19. Find each of the following to four places of decimals by means of the binomial theorem:

$$\text{(i)}\ \sqrt[2]{110},\quad \text{(ii)}\ \sqrt[3]{130},\quad \text{(iii)}\ \sqrt[4]{630}.$$

20. Find the coefficients of x^6 and x^8 in the expansion of

$$(1+x+x^2+x^3)^3.$$

21. Shew that in the expansion of $(1+x+x^2)^{-1}$ in a series of ascending powers of x, the coefficients of x^2 and x^5 are zero.

22. Shew that in the expansion of $(1+x+x^2+x^2+x^3+x^4)^{-3}$ in a series of ascending powers of x, the coefficients of x^4 and x^9 are zero.

CHAPTER XXVII.

LOGARITHMS.

256. **Definition.** The index of the power to which one number must be raised to produce a second number is called the **logarithm** of the second number with respect to the first as base. Thus if $a^x = y$, then x is called the logarithm of y to the base a, and this is expressed by the notation $x = \log_a y$.

We proceed to investigate the fundamental properties of logarithms, and to shew how logarithms can be found, and how they can be employed to shorten certain approximate calculations.

257. **Properties of Logarithms.** The following are the fundamental properties of logarithms.

I. Since $a^0 = 1$, for all values of a, it follows that $\log_a 1 = 0$.

Thus *the logarithm of* 1 *is* 0, *whatever the base may be.*

II. If $\log_a x = p$, and $\log_a y = q$;

then $x = a^p$ and $y = a^q$;

$$\therefore \quad x \times y = a^p \times x^q = a^{p+q};$$

$$\therefore \quad \log_a (xy) = p + q = \log_a x + \log_a y.$$

Similarly it can be proved that

$$\log_a (xyz \ldots) = \log_a x + \log_a y + \log_a z + \ldots.$$

Thus *the logarithm of a product is the sum of the logarithms of its factors.*

III. If $\log_a x = p$ and $\log_a y = q$;

then $x = a^p$ and $y = a^q$;

$$\therefore \quad x \div y = a^p \div a^q = a^{p-q};$$

$$\therefore \quad \log_a (x \div y) = p - q = \log_a x - \log_a y.$$

Thus *the logarithm of a quotient is the algebraic difference of the logarithms of the dividend and the divisor.*

IV. If $x = a^p$; then $x^m = a^{pm}$, for all values of m.

Hence $\log_a x^m = \log_a a^{pm} = p \times m = \log_a x \times m$.

Thus *the logarithm of any power of a number is the product of the logarithm of the number by the index of the power.*

V. Let $\log_a x = p$, and $\log_b x = q$.

Then $x = a^p$, and $x = b^q$.

Hence $a^p = b^q$, and therefore

$$a = b^{\frac{q}{p}} \text{ and } b = a^{\frac{p}{q}}.$$

Hence $\log_b a = \dfrac{q}{p}$, and $\log_a b = \dfrac{p}{q}$;

$$\therefore \quad \log_b a \times \log_a b = \frac{q}{p} \times \frac{p}{q} = 1.$$

Again, from $q = p \log_b a$,

we have $\log_b x = \log_a x \times \log_b a$.

Hence *the logarithm of any number to the base b will be found by multiplying the logarithm of that number to the base a by the constant multiplier* $\log_b a$.

Ex. 1. Find $\log_2 8$, $\log_8 2$, $\log_{10} 1000$ and $\log_{10} \sqrt[3]{100}$.

$8 = 2^3$, $2 = 8^{\frac{1}{3}}$, $1000 = 10^3$ and $\sqrt[3]{100} = 10^{\frac{2}{3}}$.

Hence $\log_2 8 = 3$, $\log_8 2 = \frac{1}{3}$, $\log_{10} 1000 = 3$ and $\log_{10} \sqrt[3]{100} = \frac{2}{3}$.

Ex. 2. Having given $\log_{10} 2 = {\cdot}3010300$ and $\log_{10} 3 = {\cdot}4771213$, find $\log_{10} 6$, $\log_{10} 40$, $\log_{10} 12$, $\log_{10} 15$ and $\log_{10} \sqrt[3]{2880}$.

$$\log_{10} 6 = \log_{10}(2\times 3) = \log_{10} 2 + \log_{10} 3$$
$$= {\cdot}3010300 + {\cdot}4771213 = {\cdot}7781513.$$

$$\log_{10} 40 = \log_{10}(4\times 10) = \log_{10} 2^2 + \log_{10} 10$$
$$= 2\times{\cdot}3010300 + 1 = 1{\cdot}6020600.$$

$$\log_{10} 12 = \log_{10}(2^2\times 3) = 2\log_{10} 2 + \log_{10} 3$$
$$= 2\times{\cdot}3010300 + {\cdot}4771213$$
$$= 1{\cdot}0791813.$$

$$\log_{10} 15 = \log_{10}\left(\frac{3\times 10}{2}\right) = \log_{10} 10 + \log_{10} 3 - \log_{10} 2$$
$$= 1 + {\cdot}4771213 - {\cdot}3010300$$
$$= 1{\cdot}1760913.$$

$$\log_{10} 288\sqrt[3]{2880} = \tfrac{1}{3}\log_{10} 0 = \tfrac{1}{3}\log_{10}(10\times 2^5\times 3^2)$$
$$= \tfrac{1}{3}(\log_{10} 10 + 5\log_{10} 2 + 2\log_{10} 3)$$
$$= \tfrac{1}{3}(1 + 1{\cdot}5051500 + {\cdot}9542426) = 1{\cdot}1531308.$$

EXAMPLES. LXXIV.

Find

1. $\log_{10} 1$.
2. $\log_{10} 10000$.
3. $\log_{10} {\cdot}01$.
4. $\log_{10} {\cdot}001$.
5. $\log_{10} \sqrt[2]{{\cdot}01}$.
6. $\log_{10} \sqrt[5]{100}$.
7. $\log_2 32$.
8. $\log_4 32$.
9. $\log_{\sqrt{2}} 2$.
10. $\log_4 8\sqrt{2}$.
11. $\log_9 3\sqrt{3}$.
12. $\log_4 \sqrt[7]{64}$.

13. Having given $\log_{10} 2 = {\cdot}3010300$ and $\log_{10} 3 = {\cdot}4771213$, find $\log_{10} \frac{2}{3}$, $\log_{10} 60$, $\log_{10} 4500$ and $\log_{10} \sqrt[5]{25}$.

14. Having given $\log_{10} 3 = {\cdot}4771213$ and $\log_{10} 5 = {\cdot}6989700$, find $\log_{10} 3{\cdot}75$, $\log_{10} 1{\cdot}28$ and $\log_{10} \frac{3^3\times 5^3}{2^7}$.

15. Having given $\log_{10} 5 = {\cdot}6989700$ and $\log_{10} 6 = {\cdot}7781513$, find $\log_{10} 324$, $\log_{10} 1{\cdot}458$ and $\log_{10} {\cdot}00432$.

16. Having given $\log_{10} 5 = {\cdot}6989700$ and $\log_{10} 7 = {\cdot}8450980$, find $\log_{10} 1{\cdot}25$, $\log_{10} 1{\cdot}28$ and $\log_{10} \frac{2^3\times 7^4}{5^5}$.

17. Given $\log_{10} 12 = 1{\cdot}0791813$ and $\log_{10} 18 = 1{\cdot}2552726$, find $\log_{10} 8$ and $\log_{10} 9$.

18. Having given $\log_{10} 24 = 1{\cdot}3802113$ and $\log_{10} 36 = 1{\cdot}5563026$, find $\log_{10} 72$ and $\log_{10} \sqrt[5]{432}$.

258. **The logarithmic series.** Let $a = e^{\lambda}$, so that

$$\lambda = \log_e a.$$

Then $$a^x = e^{\lambda x} = e^{x \log_e a}.$$

Hence, from Art. 255, we have

$$a^x = e^{x \log_e a} = 1 + x \log_e a + \frac{1}{\underline{|2}}(x \log_e a)^2 + \frac{1}{\underline{|3}}(x \log_e a)^3 + \ldots$$

Now put $a = 1 + y$; then we have

$$(1+y)^x = 1 + x \log_e (1+y) + \frac{1}{\underline{|2}}\{x \log_e (1+y)\}^2 + \ldots$$

Now, *provided* y *be numerically less than unity*, $(1+y)^x$ can be expanded by the Binomial Theorem; we then have

$$1 + xy + \frac{x(x-1)}{1 \,.\, 2} y^2 + \frac{x(x-1)(x-2)}{1 \,.\, 2 \,.\, 3} y^3 + \ldots$$

$$= 1 + x \log_e (1+y) + \frac{1}{\underline{|2}}\{x \log_e (1+y)\}^2 + \ldots.$$

Equate the coefficients of x on the two sides of the last equation. We thus obtain

$$\log_e (1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots.$$

This is called the *logarithmic series.*

259. In order to diminish the labour of finding the approximate value of the logarithm of any number, more rapidly converging series are obtained from the fundamental logarithmic series.

Changing the sign of y in the logarithmic series

$$\log_e (1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots, \qquad \ldots\ldots\text{(i)},$$

we have $$\log_e (1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \ldots. \qquad \ldots\text{(ii)}.$$

Hence $\log_e \frac{1+y}{1-y} = \log_e(1+y) - \log_e(1-y)$

[Art. 257 III.]

$$= \left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots\right) + \left(y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \dots\right);$$

$$\therefore\ \log_e \frac{1+y}{1-y} = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \dots\right) \dots\dots\dots \text{(iii)}.$$

Put $\frac{m}{n}$ for $\frac{1+y}{1-y}$, and therefore $\frac{m-n}{m+n}$ for y; then we have from (iii)

$$\log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \dots\right\} \dots \text{(iv)}.$$

We are now able to calculate logarithms to base e without much labour.

Put $m=2$, $n=1$, in formula (iv); then

$$\log_e 2 = 2\left\{\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \dots\right\}$$

from which it is easy to obtain the value

$$\log_e 2 = {\cdot}693147\dots.$$

Having found $\log_e 2$, we have from (iv)

$$\log_e \frac{3}{2} = 2\left\{\frac{1}{5} + \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} + \dots\right\};$$

$\therefore\ \log_e 3 - \log_e 2 = {\cdot}405465\dots.$

Hence $\log_e 3 = \log_e 2 + {\cdot}405465\dots$

$$= {\cdot}693147\dots + {\cdot}405465\dots = 1{\cdot}09861\dots.$$

Proceeding in this way the logarithm to base e of any number can be found to any requisite degree of approximation.

260. Logarithms to base e are called **Napierian** logarithms.

The logarithms used in all theoretical investigations are Napierian logarithms; but when approximate numerical calculations are made by means of logarithms, the logarithms used are always those to base 10, for reasons which will shortly appear; on this account logarithms to base 10 are called **Common** logarithms.

We have shewn how logarithms to base e can be found; and having found logarithms to base e, the logarithms to base 10 are obtained by multiplying by the constant factor $\log_{10}e$ or by $\frac{1}{\log_e 10}$ [Art. 257, V.]. This constant factor is called the **Modulus**: its value is $\cdot 43429\ldots$

COMMON LOGARITHMS.

261. In what follows the logarithms must always be supposed to be common logarithms, and the base 10 need not be written.

If two numbers have the same figures, and therefore differ only in the position of the decimal point, the one must be the product of the other and some integral power of 10, and hence from Art. 257, II., the logarithms of the numbers will differ by an integer.

Thus $\log 421\cdot5 = \log 4\cdot215 + \log 100 = 2 + \log 4\cdot215$.
Again, knowing that $\log 3 = \cdot30103$, we have
$$\log \cdot03 = \log(3 \div 100) = \cdot30103 - 2.$$

On account of the above property, common logarithms are always written with *the decimal part positive.* Thus $\log \cdot03$ is not written in the form $-1\cdot69897$ but $\bar{2}\cdot30103$, the minus sign referring only to the integral portion of the logarithm and being written *above* the figure to which it refers.

Definition. When a logarithm is so written that its decimal part is positive, the decimal part of the logarithm is called the **Mantissa** and the integral part the **Characteristic.**

262. *The characteristic of the logarithm of any number can be written down by inspection.*

First let the number be greater than 1, and let n be the number of figures in its integral part; then the number is clearly less than 10^n but not less than 10^{n-1}. The logarithm of the number is therefore between n and $n-1$; thus the logarithm is equal to $n-1+$ a decimal.

Thus *the characteristic of the logarithm of any number greater than unity is one less than the number of figures in its integral part.*

For example 235 is greater than 10^2 but less than 10^3. Hence $\log 235 = 2 +$ a decimal, so that the characteristic is 2.

Next, let the number be less than 1. Express the number as a decimal, and let n be the number of ciphers before its first significant figure. Then the number is less than 10^{-n} and not less than 10^{-n-1}.

Hence, *as the decimal part of the logarithm must be positive*, the logarithm of the number will be $-(n+1)+$ a decimal fraction, the characteristic being $-(n+1)$.

Thus, *if a number less than unity be expressed as a decimal, the characteristic of its logarithm is negative and greater by one than the number of ciphers before the first significant figure.*

For example, ·02 is greater than 10^{-2} but less than 10^{-1}; hence log ·02 is $-2+$ a decimal, the characteristic being -2. Also ·00042 is greater than 10^{-4} but less than 10^{-3}; hence log ·00042 is $-4+$ a decimal, the characteristic being -4.

263. Conversely, if we know the characteristic of the logarithm of any number whose digits form a certain sequence of figures, we know where to place the decimal point.

For example, knowing that log 1·1467 = ·0594498 we know that the number whose logarithm is 3·0594498 is 1146·7, and that the number whose logarithm is $\overline{4}$·0594498 is ·00011467.

264. Tables are published which give the logarithms of all numbers from 1 to 99999 calculated to seven places of decimals: these are called 'seven-figure' logarithms. For many purposes it is however sufficient to use five-figure logarithms.

In all Tables of logarithms the mantissae only are given, for the characteristics can always, as we have seen, be written down by inspection.

In making use of Tables of logarithms we have, I. to find the logarithm of a given number, and II. to find the number which has a given logarithm.

I. *To find the logarithm of a given number.*

If the number have no more than five significant figures, its logarithm will be given in the tables. But, if the number have more significant figures than are given in the tables, use must be made of the principle that when the difference of two numbers is *small* compared with either of them, the difference of the numbers is approximately proportional to the difference of their logarithms.

An example will shew how the above principle, called the *Principle of Proportional Differences*, is utilised.

Ex. 1. To find the logarithm of 1·14673.

From the tables we find that log 1·1467 = ·0594498, and that log 1·1468 = ·0594877, and the difference of these logarithms is ·0000379. Now the difference between 1·14673 and 1·1467 is $\frac{3}{10}$ths the difference between 1·1468 and 1·1467. Hence to find log 1·14673 we must add to log 1·1467 three-tenths the difference between log 1·1467 and 1·1468, that is we must add

$$\cdot 0000379 \times \tfrac{3}{10} = \cdot 0000113.$$

Hence log 1·14673 = ·0594498 + ·0000113 = ·0594611.

II. *To find the number which has a given logarithm.*

For example, to find the number whose logarithm is 2·0594611.

We find from the tables that log 1·1467 = ·0594498 and that log 1·1468 = ·0594877, the mantissa of the given logarithm falling between these two.

Now the difference between ·0594498 and the given logarithm is ·0000113, and the difference between the logarithms of 1·1467 and 1·1468 is ·0000379. Hence, by the principle of proportional differences, the number whose logarithm is ·0594611 is

$$1{\cdot}1467 + \tfrac{113}{379} \times {\cdot}0001 = 1{\cdot}1467 + {\cdot}00003 = 1{\cdot}14673.$$

[N.B. The approximation can only be relied upon for *one* figure.]

Thus ·0594611 is log 1·14673, and therefore 2·0594611 is log 114·673.

Ex. Find $\sqrt[3]{}$ having given log 4·6415 = ·6666584 and log 4·6416 = ·6666677.

$$\log \sqrt[3]{100} = \tfrac{1}{3} \log 100 = \tfrac{2}{3} = {\cdot}6666666.$$

Now $\log 4{\cdot}6416 = {\cdot}6666677,$

$\log 4{\cdot}6415 = {\cdot}6666584.$

Hence $\log \sqrt[3]{100} - \log 4{\cdot}6415 = {\cdot}0000082,$

and $\log 4{\cdot}6416 - \log 4{\cdot}6415 = {\cdot}0000093.$

Hence $\sqrt[3]{100} = 4{\cdot}6415 + \tfrac{82}{93}$ of $\cdot0001 = 4{\cdot}64159.$

COMPOUND INTEREST AND ANNUITIES.

265. The approximate calculation of Compound Interest for a long period, and also of the value of an annuity, can be readily found by means of logarithms.

All problems of this kind depend upon the three following. [The student is supposed to be acquainted with the arithmetical treatment of interest and present worth.]

I. *To find the amount of a given sum at Compound Interest, in a given number of years and at a given rate per cent.*

Let P denote the principal, n the number of years, £r the interest of £1 for 1 year, and A the required amount.

Then the interest of P for 1 year will be Pr, and therefore the amount of principal and interest at the end of the first year will be $P(1+r)$. This last sum is the capital on which interest is to be paid for the second year; and therefore the amount at the end of the second year will be $[P(1+r)](1+r)=P(1+r)^2$. Similarly the amount at the end of 3 years will be $P(1+r)^3$, and at the end of n years will be $P(1+r)^n$.

Thus $$A=P(1+r)^n.$$

Cor. If the interest be paid and capitalised half yearly, it can be easily seen that the amount at the end of n years will be $P\left(1+\frac{r}{2}\right)^{2n}$.

Ex. Find the amount of £100 in 25 years at 5 per cent. per annum.

Since the interest on £100 is £5, the interest on £1 is £$\frac{1}{20}$. Hence the amount of £1 at the end of the first year will be £$(1+\frac{1}{20})$, and at the end of 25 years will be £$(1+\frac{1}{20})^{25}$.

Hence £A, the amount required, will be £$100\ (1+\frac{1}{20})^{25}$.

Hence $\log A=\log 100+25\log\frac{21}{20}$.

From the tables we find that

$$\log 21=1\cdot3222193, \text{ and } \log 20=1\cdot3010300.$$

Hence $\log A=2+25\,(1\cdot3222193-1\cdot3010300)$

$=2\cdot5297325.$

Now from the tables we find $\log 338\cdot63=2\cdot5297254$

and $\log 338\cdot64=2\cdot5297383.$

Hence $\log A-\log 338\cdot63=\cdot0000071,$

and $\log 338\cdot64-\log 338\cdot63=\cdot0000129.$

Hence $A=338\cdot63+\frac{71}{129}$ of $\cdot01$

$=338\cdot63+\cdot005=338\cdot635.$

Hence the amount required is £338·635.

II. *To find the present value of a sum of money which is to be paid at the end of a given time.*

Let A be the sum payable at the end of n years, and let P be its present worth, the interest on £1 being £r per annum. Then the amount of P in n years must be just equal to A. $P=\frac{A}{(1+r)^n}$

Hence, from I. $P=A(1+r)^{-n}$.

Ex. What is the present worth of £1000 payable at the end of 100 years, interest being at the rate of 5 per cent. per annum?

$$P=1000\,(1+\tfrac{1}{20})^{-100}=1000\,(\tfrac{21}{20})^{-100}.$$

Hence $\log P=\log 1000-100\,(\log 21-\log 20)$

$$=3-100\,(1{\cdot}3222193-1{\cdot}3010300)$$

$$=3-2{\cdot}11893={\cdot}88107.$$

Now from the tables $\log 7{\cdot}6045={\cdot}8810707$.

Hence the required present worth is £7·6045 = £7. 12*s.* 1*d.*

III. *To find the present value of an annuity of* £A *payable at the end of each of* n *successive years.*

If the interest on £1 be £r; then from II.

the present value of the first payment is $A\,(1+r)^{-1}$,

..........................second........... $A\,(1+r)^{-2}$,

....

.... nth .... $A\,(1+r)^{-n}$.

Hence the present value of the whole is

$$A\,\{(1+r)^{-1}+(1+r)^{-2}+\ldots+(1+r)^{-n}\}$$

$$=A\,(1+r)^{-1}\,\frac{1-(1+r)^{-n}}{1-(1+r)^{-1}}=\frac{A}{r}\{1-(1+r)^{-n}\}.$$

Ex. Find the present value of an annuity of £100 to be paid for 30 years, reckoning interest at 4 per cent.

The interest on £1 is £$\frac{4}{100}$, therefore $1+r=1{\cdot}04$.

Hence $P=\dfrac{100}{\frac{4}{100}}\{1-(1{\cdot}04)^{-30}\}$

$$=2500\,\{1-(1{\cdot}04)^{-30}\}.$$

Now $\log(1{\cdot}04)^{-30}=-30\log(1{\cdot}04)=-30\times{\cdot}0170333$

$$=-{\cdot}510999=\bar{1}{\cdot}489001.$$

From the tables $\log 3{\cdot}0832={\cdot}4890017$, hence $\bar{1}{\cdot}489001$ is log ·30832.

Hence $P=2500\,(1-{\cdot}30832)=2500\times{\cdot}69168$

$$=£1729{\cdot}2.$$

EXAMPLES. LXXV.

1. Write down the numbers whose logarithms are 5·3010300 and $\bar{5}\cdot3010300$ respectively, knowing that log 2 = ·3010300.

2. Write down the numbers whose logarithms are 3·2990931 and $\bar{4}\cdot2990931$ respectively, knowing that log 1·9911 = ·2990931.

3. Having given log 46854 = 4·6707467 and log 46855 = 4·6707559, find log ·0468546.

4. Having given log 58961 = 4·7705648 and 58·962 = 1·7705722, find log ·00589614.

5. Find $\sqrt[5]{29}$, having given log 29 = 1·4623980,
log 19611 = 4·2924776, and log 19612 = 4·2924997.

6. Find $\sqrt[7]{100}$, having given log 19307 = ·2857148, and
log 19308 = ·2857373.

7. Having given log 2 = ·3010300, log 3 = ·4771213,
log 17187 = 4·2352001 and log 17188 = 4·2352253,
find $\sqrt[5]{15}$ to 5 places of decimals.

8. Having given log 3·4277 = ·5350028, log 32483 = 4·5116561 and log 32484 = 4·5116695, find $\sqrt[3]{\cdot034277}$ to 6 places of decimals.

9. Find the amount of £1 in 100 years at 5 per cent. compound interest, having given log 1·05 = ·0211893, log 1·3150 = ·1189258, and log 1·3151 = ·1189588.

10. Shew that a sum of money will be more than doubled in 18 years at 4 per cent. compound interest, having given log 1·04 = ·0170333 and log 2 = ·3010300.

11. Find the amount of £500 in 10 years at 4 per cent. compound interest, the interest being payable half-yearly. Given
log 1·02 = ·0086002, log 1·4859 = ·1719896 and log 1·4860 = ·1720188.

12. What is the present value of £1000 which is to be paid at the end of 15 years, reckoning compound interest at 3 per cent.? Given log 1·03 = ·0128372, log 64186 = 4·8074403 and log 64187 = 4·8074471.

CHAPTER XXVIII.

MISCELLANEOUS THEOREMS AND EXAMPLES. CUBE ROOT.

266. **Theorem.** *The expression $x^n - a^n$ is divisible by $x - a$ for all positive integral values of n.*

It is known that $x - a$, $x^2 - a^2$ and $x^3 - a^3$ are all divisible by $x - a$.

We have
$$\begin{aligned} x^n - a^n &= x^n - ax^{n-1} + ax^{n-1} - a^n \\ &= x^{n-1}(x - a) + a(x^{n-1} - a^{n-1}). \end{aligned}$$

Now, *if* $x - a$ divides $x^{n-1} - a^{n-1}$ it will also divide
$$x^{n-1}(x - a) + a(x^{n-1} - a^{n-1}),$$
that is, it will divide $x^n - a^n$.

Hence, *if* $x - a$ divides $x^{n-1} - a^{n-1}$ it will also divide $x^n - a^n$.

But we know that $x - a$ divides $x^3 - a^3$; it will therefore also divide $x^4 - a^4$. And, since $x - a$ divides $x^4 - a^4$ it will also divide $x^5 - a^5$. And so on indefinitely.

Hence $x^n - a^n$ *is divisible by* $x - a$, *when* n *is any positive integer.*

Since $x^n + a^n = x^n - a^n + 2a^n$, it follows that when $x^n + a^n$ is divided by $x - a$ the remainder is $2a^n$, so that $x^n + a^n$ is never divisible by $x - a$.

If we change a into $-a$, $x - a$ becomes $x - (-a) = x + a$; also $x^n - a^n$ becomes $x^n - (-a)^n$, and $x^n - (-a)^n$ is $x^n + a^n$ or $x^n - a^n$ according as n is odd or even.

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Hence, when n is *odd*

$$x^n + a^n \text{ is divisible by } x + a,$$

and when n is *even*

$$x^n - a^n \text{ is divisible by } x + a.$$

Thus, when n is a positive integer

$x - a$ divides $x^n - a^n$ always,

$x - a$ .... $x^n + a^n$ never,

$x + a$ .... $x^n - a^n$ when n is even,

and $x + a$ .... $x^n + a^n$ .... odd.

We have in the above shewn that the four cases are all included in the first: we leave it as an exercise for the student to prove each case separately.

The above results may be written so as to shew the quotients: thus

$$\frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + a^{n-1},$$

$$\frac{x^n \pm a^n}{x + a} = x^{n-1} - x^{n-2}a + x^{n-3}a^2 - \dots \pm a^{n-1},$$

the upper or lower signs being taken on each side of the second formula according as n is odd or even.

EXAMPLES. LXXVI.

1. Find the factors of $x^3 - x^2 - x + 1$.
2. Find the factors of $x^6 - x^4 - x^2 + 1$.
3. Find the factors of $1 - x - x^4 + x^5$.
4. Find the factors of $1 - x^2 - x^8 + x^{10}$.
5. Find the factors of $x^2 - y^2 - 2ax - 2by + a^2 - b^2$.
6. Find the factors of $x^2 - y^2 - 3x - y + 2$.
7. Find the factors of $x^2 - 2xy - 8y^2 - 2x + 20y - 8$.

7 4

8. Find the factors of $a^2-b^2+c^2-f^2-2ac-2bf$.

✓ **9.** Write down the quotient in each of the following divisions:

(i) $(x^5-y^5)\div(x-y)$, (ii) $(x^7+y^7)\div(x+y)$, (iii) $(x^8-y^8)\div(x+y)$.

10. Shew that, if n is any positive integer, 7^n-1 is divisible by 6; shew also that $35^{2n+1}+1$ is divisible by 36.

11. Shew without actual division that

$$(3x^2-2x+1)^3-(2x^2+3x-5)^3$$

is divisible by x^2-5x+6.

12. *Write down* the result of dividing $(2a+3b)^3+(3a+2b)^3$ by $5a+5b$.

13. *Write down* the result of dividing $(2a+4b-4c)^3+(a-b+7c)^3$ by $a+b+c$.

14. Shew that $(1-x)^2$ is a factor of $1-x-x^5+x^6$.

15. Shew that $(1-x)^2$ is a factor of $1-x-x^n+x^{n+1}$, n being any positive integer.

16. Shew that $(x-1)^2$ is a factor of $nx^{n+1}-(n+1)x^n+1$, and also of $x^n-nx+n-1$, n being any positive integer.

267. We now prove the following important proposition:

Theorem. *If any rational and integral expression which contains x vanish when f is put for x, then will $x-f$ be a factor of the expression.*

Let the expression, arranged according to powers of x, be

$$ax^n+bx^{n-1}+cx^{n-2}+\ldots$$

Then, by supposition,

$$af^n+bf^{n-1}+cf^{n-2}+\ldots=0.$$

Hence $\quad ax^n+bx^{n-1}+cx^{n-2}+\ldots$

$$=ax^n+bx^{n-1}+cx^{n-2}+\ldots-(af^n+bf^{n-1}+cf^{n-2}+\ldots)$$

$$=a(x^n-f^n)+b(x^{n-1}-f^{n-1})+c(x^{n-2}-f^{n-2})+\ldots$$

But, by the last Article, x^n-f^n, $x^{n-1}-f^{n-1}$, $x^{n-2}-f^{n-2}$, &c. are all divisible by $x-f$.

Hence also $ax^n+bx^{n-1}+cx^{n-2}+\ldots$ is divisible by $x-f$.

268. The proposition proved in the preceding Article can also be proved in the following manner:

Divide the expression $ax^n + bx^{n-1} + cx^{n-2} + \dots$ by $x-f$, continuing the process until the remainder, if there be any remainder, does not contain x; and let Q be the quotient and R the remainder.

Then, by the nature of division,

$$ax^n + bx^{n-1} + cx^{n-2} + \dots = Q(x-f) + R,$$

and the two members of this equation are *identical*.

Now since R does not contain x, no change will be made in R by changing the value of x: put then $x=f$, and we have

$$af^n + bf^{n-1} + cf^{n-2} + \dots = Q(f-f) + R = R.$$

Hence *if any expression which contains x be divided by $x-f$, the remainder is equal to the result obtained by putting f in the place of x in the expression.*

This includes as a particular case the proposition that $x-f$ is a factor of any expression which vanishes when f is put for x.

Ex. 1. Find the remainder when x^3-4x^2+2x+1 is divided by $x-3$.

The remainder is $3^3-4.3^2+2.3+1=-2$.

Ex. 2. Shew that $x-2$ is a factor of x^4-3x^2+2x-8.

The condition is $2^4-3.2^2+2.2-8=0$; and this condition is satisfied.

Ex. 3. Find the L.C.M. of

$$x^2-3x+2, \text{ and } x^3-13x+12.$$

The factors of x^2-3x+2 are obviously $x-1$ and $x-2$; and of these $x-1$ does, and $x-2$ does not satisfy the above condition of being a factor of $x^3-13x+12$. Thus the L.C.M. is

$$(x-2)(x^3-13x+12).$$

Ex. 4. Shew that any rational and integral expression is divisible by $x-1$ if the sum of the coefficients of the different powers of x is zero.

For if the sum of the coefficients is zero, the expression will vanish when we put $x=1$, and therefore $x-1$ is a factor.

269. We have proved that $x-\alpha$ is a factor of the expression $ax^n+bx^{n-1}+cx^{n-2}+\ldots$, provided that the expression vanishes when α is put for x.

If the division were actually performed it is clear that the first term of the quotient, which is the term of the highest degree in x, would be ax^{n-1}.

Hence the given expression is equivalent to

$$(x-\alpha)(ax^{n-1}+\text{\&c.}).$$

Now suppose that the given expression also vanishes when $x=\beta$; then the product of $x-\alpha$ and $ax^{n-1}+\text{\&c.}$ will vanish when $x=\beta$; and, since $x-\alpha$ does not vanish when $x=\beta$, it follows that $ax^{n-1}+\text{\&c.}$ must vanish when $x=\beta$. Hence $x-\beta$ is a factor of $ax^{n-1}+\text{\&c.}$; and, if the division were performed, it is clear that the first term of the quotient would be ax^{n-2}.

Hence the original expression is equivalent to

$$(x-\alpha)(x-\beta)(ax^{n-2}+\text{\&c.}\ldots)$$

Similarly, if the original expression vanishes also for the values γ, δ, &c. of x, it is equivalent to

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\ldots(ax^{n-r}+\text{\&c.}),$$

where r is equal to the number of the factors $x-\alpha$, $x-\beta$, &c.

If therefore the given expression vanishes for n values α, β, γ, &c., there are n factors such as $x-\alpha$, and the remaining factor, $ax^{n-r}+\text{\&c.}$, will reduce to a; and hence the given expression is equivalent to

$$a(x-\alpha)(x-\beta)(x-\gamma)\ldots\ldots$$

270. **Theorem.** *An expression of the* n^{th} *degree in* x *cannot vanish for more than* n *values of* x.

For, if the expression

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots$$

vanish for the n values $\alpha, \beta, \gamma \ldots$, it must be equivalent to

$$a(x-\alpha)(x-\beta)(x-\gamma)\ldots.$$

If now we substitute any other value, k suppose, different from each of the n values α, β, γ, &c.; then, since no one of the factors $k-\alpha$, $k-\beta$, &c. is zero, their continued product cannot be zero, and therefore the given expression cannot vanish for the value $x = k$, *except* a *itself is zero.*

But, if a is zero, the original expression reduces to $bx^{n-1} + cx^{n-2} + \ldots$, and is of the $(n-1)^{\text{th}}$ degree; and hence it can only vanish for $n-1$ values of x, except b is zero. And so on.

Hence *an expression of the* n^{th} *degree in* x *cannot vanish for more than* n *values of* x, *except the coefficients of all the powers of* x *are zero;* and when all these coefficients are zero, the expression will clearly vanish for any value whatever of x.

271. The values of x for which the *expression*

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots$$

is equal to zero, are the roots of the *equation*

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots = 0.$$

Hence, by Art. 270, *an equation of the* n^{th} *degree cannot have more than* n *roots,* except the coefficients of all the different powers of the unknown quantity are zero, in which case any value of x satisfies the equation.

272. If the two expressions of the n^{th} degree

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots,$$

and $$px^n + qx^{n-1} + rx^{n-2} + \ldots,$$

In a [illegible], i.e. [illegible]

[illegible] = 0

be equal to one another for more than n values of x, it follows that the equation

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots = px^n + qx^{n-1} + rx^{n-2} + \ldots,$$

that is, the equation

$$(a-p)\,x^n + (b-q)\,x^{n-1} + (c-r)\,x^{n-2} + \ldots = 0$$

has more than n roots.

Hence, by Art. 271, the coefficients of all the different powers of x must be equal to zero.

Thus $\qquad a = p, \quad b = q, \quad c = r, \quad$ &c.

Hence, *if two expressions of the* n^{th} *degree in* x *are equal to one another for more than* n *values of* x, *the coefficient of any power of* x *in one expression is equal to the coefficient of the same power of* x *in the other expression.*

273. When any two expressions, which have a limited number of terms, are equal to one another for *all values*, the condition of the last Article is clearly satisfied, for the number of values must be greater than the index of the highest power of any contained letter.

Hence *when any two expressions, which have a limited number of terms, are equal to one another for all values of the letters involved in them, we may equate the coefficients of the different powers of any letter.*

We may remark that the above proposition is true of two expressions which have an infinite number of terms, but the consideration of such expressions is beyond the range of this book.

274. **Symmetrical expressions.** An expression is said to be *symmetrical* with respect to any two letters when the expression is unaltered by interchanging the two letters.

Thus $a + b$ and $a^2 + b^2$ are symmetrical with respect to

a and b, for neither expression would be altered by changing a into b and b into a.

Also $a+b+c$ and $a^3+b^3+c^3-3abc$ are symmetrical with respect to any two of the three letters a, b, c.

It is clear that the only expression of the first degree which is symmetrical with respect to the three letters a, b, c is $pa+pb+pc$, where p is some numerical coefficient.

275. **Cyclical order.** It is of importance at this stage for the student to attend to the way in which expressions are usually arranged. Consider, for example, the arrangement of the expression $bc+ca+ab$. The term which does not contain the letter a is put first, and the other terms can be obtained in succession by a *cyclical change of the letters*, that is by changing a into b, b into c and c into a. In the expression $a^2(b-c)+b^2(c-a)+c^2(a-b)$ the same arrangement is observed; for by making a cyclical change in the letters of $a^2(b-c)$ we obtain $b^2(c-a)$, and another cyclical change will give $c^2(a-b)$. So also the second and third factors of $(b-c)(c-a)(a-b)$ are obtained from the first by cyclical changes.

276. We now proceed to consider some examples which will illustrate the theorems proved in the preceding articles.

Ex. 1. Shew that

$$(b-c)^5+(c-a)^5+(a-b)^5$$

is divisible by $\qquad (b-c)(c-a)(a-b)$.

If we put $b=c$ in the expression

$$(b-c)^5+(c-a)^5+(a-b)^5$$

the result is $\qquad (c-a)^5+(a-c)^5$,

which is zero.

Hence $(b-c)$ is a factor, and we can prove in a similar manner that $c-a$ and $a-b$ are factors.

Ex. 2. Shew that

$$(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3=24abc.$$

If we put $a=0$ in the expression

$$(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3\text{.........(i)},$$

it is easy to see that the result will vanish for all values of b and c.

Hence a is a factor of (i).

So also b and c are factors of (i).

Now (i) is an expression of the *third* degree; it can therefore only have *three* factors. Hence it is either equal to abc, or is equal to abc multiplied by some *number*.

Thus

$$(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3=Labc\text{...(ii)},$$

where L is some number which is always the same whatever a, b and c may be.

We can find the value of L by giving particular values to a, b, and c.

Thus, let $a=b=c=1$. Then (ii) becomes

$$3^3-1^3-1^3-1^3=L;$$

$$\therefore\ L=24.$$

We can also find the value of L by actually finding the coefficient of abc in the left member of (ii). By Art. 251 the coefficient of abc in $(a+b+c)^3$ is $\dfrac{\underline{|3}}{\underline{|1}\,\underline{|1}\,\underline{|1}}=6$, and similarly the coefficient of abc in each of the other cubes is -6: thus the whole coefficient is 24. The coefficient on the right is L; hence, as before, $L=24$.

Ex. 3. Find the factors of

$$a^3(\overset{c}{b}-c)+\overset{c^3}{b^3}(c-a)+c^3(a-\overset{a}{b}).$$

If we put $b=c$ in the expression

$$a^3(b-c)+b^3(c-a)+c^3(a-b),$$

the result is $\qquad c^3(c-a)+c^3(a-c),$

which is clearly zero.

Hence $b-c$ is a factor of the given expression; and we can prove in a similar manner that $c-a$ and $a-b$ are also factors.

Now the given expression is of the *fourth* degree; hence, besides the three factors we have found, there must be one other factor of the *first* degree, and as this factor must be symmetrical in a, b, c it must be $a+b+c$.

Hence the given expression must be equal to

$$L(b-c)(c-a)(a-b)(a+b+c),$$

where L is a number.

We can find L by giving particular values to a, b, and c; or, by comparing the coefficients of a^3, we have at once

$$b-c=-L(b-c).$$

Hence $L=-1$.

Thus

$$a^3(b-c)+b^3(c-a)+c^3(a-b)=-(b-c)(c-a)(a-b)(a+b+c).$$

Ex. 4. Simplify

$$\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}.$$

The L.C.M. of the denominators is $(b-c)(c-a)(a-b)$.

Hence the given expression is equal to

$$\frac{-a^2(b-c)-b^2(c-a)-c^2(a-b)}{(b-c)(c-a)(a-b)}. \quad b-c$$

Now we naturally test whether either of the factors of the denominator is also a factor of the numerator: we are thus led to find that the numerator is the same as the denominator, so that the given expression is equal to unity.

277. The following is an important identity:

$$\underline{a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab).} \quad \text{K. B.}$$

It should be noticed that

$$a^2+b^2+c^2-bc-ca-ab=\tfrac{1}{2}\{(b-c)^2+(c-a)^2+(a-b)^2\}.$$

Since $a+b+c$ is a factor of $a^3+b^3+c^3-3abc$, it follows that $a^3+b^3+c^3-3abc=0$ if $a+b+c=0$.

Hence $a^3+b^3+c^3=3abc$ for all values of a, b and c, provided only that $a+b+c=0$.

That is, *the sum of the cubes of any three quantities is equal to three times their product, provided that the sum of the three quantities is zero.*

For example,

$$(b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b),$$

$$(b+c-2a)^3+(c+a-2b)^3+(a+b-2c)^3$$
$$=3(b+c-2a)(c+a-2b)(a+b-2c);$$

also, since

$$(x-a)(b-c)+(x-b)(c-a)+(x-c)(a-b)=0,$$

we have

$$(x-a)^3(b-c)^3+(x-b)^3(c-a)^3+(x-c)^3(a-b)^3$$
$$=3(x-a)(x-b)(x-c)(b-c)(c-a)(a-b).$$

278. We have seen that we must have more than one equation in order to determine the values of two or more unknown quantities. This however is by no means always the case when the values of the unknown quantities are in any way restricted.

For example, if we have the single equation $3x+4y=10$, and restrict both x and y to positive integral values, the equation can only be satisfied by one set of values, namely by the values $x=2$, $y=1$.

Again, from the relation

$$(x-a)^2+(y-b)^2=0,$$

with the restriction that all the quantities must be *real*, we can conclude both that $x=a$ and that $y=b$; for the squares of real quantities must be positive, and the sum of two positive quantities cannot be zero unless they are both zero.

An important example is considered in Art. 190.

Ex. Shew that, if a, b, c, d are all real and

$$(a+b)^2+(b+c)^2+(c+d)^2=4(ab+bc+cd),$$

then will $$a=b=c=d.$$

We must write the expression

$$(a+b)^2+(b+c)^2+(c+d)^2-4(ab+bc+cd)$$

as the sum of two or more squares: we then have

$$(a-b)^2+(b-c)^2+(c-d)^2=0.$$

Then, since $a-b$, $b-c$, and $c-d$, are all real, we must have

$$a=b,\ b=c,\ \text{and}\ c=d.$$

279. The following are important examples.

Ex. 1. Find the greatest value of the product of two positive quantities whose sum is given.

Let the given sum be $2a$. Then, if one of the quantities be $a+x$, the other will be $a-x$; also the product will be a^2-x^2.

Now, since a is fixed, a^2-x^2 has its greatest value when $x=0$, and in that case the two quantities are equal to one another.

Thus, *if the sum of two positive quantities is given, their product is greatest when they are equal to one another.*

Ex. 2. Find the least value of the sum of two positive quantities whose product is given.

Since $(x+y)^2 = 4xy + (x-y)^2$, we see that $(x+y)^2$ is never less than $4xy$, and that $(x+y)^2$ is equal to $4xy$ when $x=y$.

Hence, *if the product of two positive quantities is given, their sum is least when they are equal to one another.*

Ex. 3. The arithmetic mean of any two positive quantities is greater than their geometric mean.

Let the two quantities be a and b.

The arithmetic mean is $\frac{1}{2}(a+b)$, and the geometric mean is $\sqrt{ab}$.

We have therefore to prove that

$$\tfrac{1}{2}(a+b) > \sqrt{ab},$$

or that

$$(a+b)^2 > 4ab,$$

or that

$$(a-b)^2 > 0,$$

which is the case, since $(a-b)^2$ is necessarily *positive.*

CUBE ROOT.

280. We now shew how to find the cube root of any algebraical expression.

We know that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Thus the cube of a binomial expression has *four* terms; also, when the cube is arranged according to powers of some letter, the cube roots of its extreme terms are the terms of the original binomial.

Conversely, if an expression which is a perfect cube contains four terms arranged according to powers of some letter, its cube root is the sum of the cube roots of the extreme terms.

Thus the cube root of any perfect cube which has only four terms can be written down by inspection.

For example, to find the cube root of

$$27a^6 - 54a^5b + 36a^4b^2 - 8a^3b^3.$$

The cube roots of the extreme terms are $3a^2$ and $-2ab$. Hence the required cube root is $3a^2 - 2ab$.

Again, the cube root of $a^9 - 6a^6b^2 + 12a^3b^4 - 8b^6$ is $a^3 - 2b^2$.

281. When an expression which contains only three different powers of a particular letter is arranged according to powers of that letter, there will only be four terms. For example, the expression

$$a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3ab^2 + 3ac^2 + 6abc + 3b^2c + 3bc^2,$$

which contains only three different powers of a, namely a, a^2 and a^3, when arranged according to powers of a, is

$$a^3 + 3a^2(b+c) + 3a(b^2+c^2+2bc) + (b^3+c^3+3b^2c+3bc^2),$$

that is $$a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3.$$

This expression therefore comes under the case considered in the previous Article; and its cube root is at once seen to be $a+b+c$.

282. We will now consider the most general case.

Suppose we have to find the cube root of $(A+B)^3$, where

A stands for any number of terms of the root, and B for the rest; the terms in A and B being arranged according to descending (or ascending) powers of some letter, so that every term in A is of higher (or lower) degree in that letter than any term in B.

Also suppose that the terms in A are known, and that we have to find the terms in B.

Subtracting A^3 from $(A+B)^3$, we have the remainder

$$(3A^2+3AB+B^2)\,B.$$

Now from the mode of arrangement it follows that the term of the highest (or lowest) degree in the remainder is 3 × square of the first term of A × first term of B.

Hence to obtain the *next term* of the required root, that is, to obtain the highest (or lowest) term of B, we *subtract from the whole expression the cube of that part of the root which is already found, and divide the first term of the remainder by three times the square of the first term of the root.*

This gives a method of finding the successive terms of the root after the first; and the first term of the root is clearly the cube root of the first term of the given expression.

Find the cube root of

$$x^6-6x^5y+21x^4y^2-44x^3y^3+63x^2y^4-54xy^5+27y^6.$$

$$\begin{array}{rl}
 & x^6-6x^5y+21x^4y^2-44x^3y^3+63x^2y^4-54xy^5+27y^6. \\
(x^2)^3 = & x^6 \\
\hline
(x^2-2xy)^3 = & x^6-6x^5y+12x^4y^2-8x^3y^3 \\
\hline
(x^2-2xy+3y^2)^3 = & x^6-6x^5y+21x^4y^2-44x^3y^3+63x^2y^4-54xy^5+27y^6.
\end{array}$$

We first take the cube root of the first term of the given expression: we thus obtain x^2, the *first term* of the required root.

We then subtract the cube of x^2 from the given expression, and divide the first term of the remainder, namely $-6x^5y$, by $3\times(x^2)^2$: we thus obtain $-2xy$, the *second term* of the root.

We then subtract the cube of $x^2 - 2xy$ from the given expression, and divide the first term of the remainder, namely $9x^4y^2$, by $3 \times (x^2)^2$: we thus obtain $3y^2$, the *third term* of the root.

Now subtract the cube of $x^2 - 2xy + 3y^2$ from the given expression and there is no remainder. Hence the given expression is $(x^2 - 2xy + 3y^2)^3$, so that the required root is $x^2 - 2xy + 3y^2$.

The cubes of x^2, $x^2 - 2xy$, &c. are placed under the given expression, like terms being placed in the same column: the remainder left after taking away any cube is then obvious.

It should be remarked that instead of finding each cube independently some labour can be saved by making use of the previous cube; the saving of labour is not however sufficiently great to make this a matter of much importance.

283. The *fourth* root of an expression can be obtained by taking the square root of its square root; also the *sixth* root by taking the square root of its cube root.

There is no difficulty in finding any assigned root of an algebraical expression.

The following rule, which depends upon the Binomial Theorem, may be given for finding the nth root of any expression:—

Arrange the expression according to descending powers of some letter, and take the nth root of the first term: this gives the *first term* of the required root.

Also, having found any number of terms of the root, subtract from the given expression the nth power of that part of the root which is already found, and divide the first term of the remainder by n times the $(n-1)$th power of the first term of the root: this gives the *next term* of the root.

Note. Roots higher than the second are very rarely required except such as can easily be seen by inspection.

EXAMPLES. LXXVII.

1. Shew that $a^2(b-c)+b^2(c-a)+c^2(a-b)$
$=bc(b-c)+ca(c-a)+ab(a-b)=-(b-c)(c-a)(a-b)$.

2. Shew that $a^4(b-c)+b^4(c-a)+c^4(a-b)$
$=-(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)$.

3. Shew that $(b-c)^5+(c-a)^5+(a-b)^5$
$=5(b-c)(c-a)(a-b)(a^2+b^2+c^2-bc-ca-ab)$.

4. Shew that
$(a+b+c)^4-(b+c)^4-(c+a)^4-(a+b)^4+a^4+b^4+c^4=12abc(a+b+c)$.

5. Shew that $a(b-c)^3+b(c-a)^3+c(a-b)^3$
$=(b-c)(c-a)(a-b)(a+b+c)$.

6. Shew that $b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)$
$=-(b-c)(c-a)(a-b)(bc+ca+ab)$.

7. Shew that
$a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)=-(b-c)(c-a)(a-b)(bc+ca+ab)$.

8. Shew that $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$
$=-(b+c)(c+a)(a+b)(b-c)(c-a)(a-b)$.

9. Find the factors of $bc(b^2-c^2)+ca(c^2-a^2)+ab(a^2-b^2)$.

10. Find the factors of $a(b+c-a)^2+b(c+a-b)^2+c(a+b-c)^2$
$+(b+c-a)(c+a-b)(a+b-c)$.

11. Find the factors of
$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)-(b+c-a)(c+a-b)(a+b-c)$.

12. Find the factors of $a(b+c)(b^2+c^2-a^2)$
$+b(c+a)(c^2+a^2-b^2)+c(a+b)(a^2+b^2-c^2)$.

13. Shew that $(x+y+z)^{2n+1}-x^{2n+1}-y^{2n+1}-z^{2n+1}$ is divisible by $(y+z)(z+x)(x+y)$, and find the quotient when $n=1$, and when $n=2$.

14. Shew that $(x+y)^3-x^3-y^3=3xy(x+y)$, and that
$(x+y)^5-x^5-y^5=5xy(x+y)(x^2+xy+y^2)$.

15. Simplify $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-c)(b-a)}+\dfrac{c^3}{(c-a)(c-b)}$.

16. Simplify $\dfrac{bc}{(a-b)(a-c)}+\dfrac{ca}{(b-c)(b-a)}+\dfrac{ab}{(c-a)(c-b)}$.

17. Simplify $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-c)(b-a)}+\dfrac{1}{c(c-a)(c-b)}$.

18. Simplify $\frac{1}{a^2(a-b)(a-c)}+\frac{1}{b^2(b-c)(b-a)}+\frac{1}{c^2(c-a)(c-b)}$.

19. Simplify $\frac{b+c}{(a-b)(a-c)}+\frac{c+a}{(b-c)(b-a)}+\frac{a+b}{(c-a)(c-b)}$.

20. Simplify $\frac{b+c}{a(a-b)(a-c)}+\frac{c+a}{b(b-c)(b-a)}+\frac{a+b}{c(c-a)(c-b)}$.

21. Shew that $\frac{1}{(a-b)(a-c)(x+a)}+\frac{1}{(b-c)(b-a)(x+b)}$

$$+\frac{1}{(c-a)(c-b)(x+c)}=\frac{1}{(x+a)(x+b)(x+c)}.$$

22. Shew that $\frac{a}{(a-b)(a-c)(x+a)}+\frac{b}{(b-c)(b-a)(x+b)}$

$$+\frac{c}{(c-a)(c-b)(x+c)}=\frac{-x}{(x+a)(x+b)(x+c)}.$$

23. Shew that $\frac{a^2}{(a-b)(a-c)(x+a)}+\frac{b^2}{(b-c)(b-a)(x+b)}$

$$+\frac{c^2}{(c-a)(c-b)(x+c)}=\frac{x^2}{(x+a)(x+b)(x+c)}.$$

24. Simplify $\frac{(b^2-c^2)^3+(c^2-a^2)^3+(a^2-b^2)^3}{(b-c)^3+(c-a)^3+(a-b)^3}$.

25. Prove the following:

(i) $a+b=\frac{a^2}{a-b}+\frac{b^2}{b-a}$.

(ii) $a+b+c=\frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-c)(b-a)}+\frac{c^3}{(c-a)(c-b)}$.

(iii) $a+b+c+d=\frac{a^4}{(a-b)(a-c)(a-d)}+\frac{b^4}{(b-c)(b-d)(b-a)}$

$$+\frac{c^4}{(c-d)(c-a)(c-b)}+\frac{d^4}{(d-a)(d-b)(d-c)}.$$

26. Shew that $\frac{a^3}{(a-b)(a-c)(a-d)}+\frac{b^3}{(b-c)(b-d)(b-a)}$

$$+\frac{c^3}{(c-d)(c-a)(c-b)}+\frac{d^3}{(d-a)(d-b)(d-c)}=1.$$

27. Shew that $\frac{bcd}{(a-b)(a-c)(a-d)}+\frac{cda}{(b-c)(b-d)(b-a)}$

$$+\frac{dab}{(c-d)(c-a)(c-b)}+\frac{abc}{(d-a)(d-b)(d-c)}=-1.$$

28. Shew that $\dfrac{bc}{(a-b)(a-c)}(x-a)^2+\dfrac{ca}{(b-c)(b-a)}(x-b)^2$

$$+\frac{ab}{(c-a)(c-b)}(x-c)^2=x^2.$$

29. Prove that $(b+c)^2+(c+a)^2+(a+b)^2-(c+a)(a+b)$
$-(a+b)(b+c)-(b+c)(c+a)=a^2+b^2+c^2-bc-ca-ab.$

30. Shew that $(b+c)^3+(c+a)^3+(a+b)^3-3(b+c)(c+a)(a+b)$
$=2(a^3+b^3+c^3-3abc).$

31. Shew that $(b+c-a)^3+(c+a-b)^3$
$+(a+b-c)^3-3(b+c-a)(c+a-b)(a+b-c)=4(a^3+b^3+c^3-3abc).$

32. Shew that $(x^2+2yz)^3+(y^2+2zx)^3$
$+(z^2+2xy)^3-3(x^2+2yz)(y^2+2zx)(z^2+2xy)=(x^3+y^3+z^3-3xyz)^2.$

33. Shew that, if $x^2+y^2+z^2=(yz+zx+xy)$, then $x=y=z$.

34. Prove the following:

(i) if $2(a^2+b^2)=(a+b)^2$, then $a=b$;

(ii) if $3(a^2+b^2+c^2)=(a+b+c)^2$, then $a=b=c$;

(iii) if $4(a^2+b^2+c^2+d^2)=(a+b+c+d)^2$, then $a=b=c=d$;

and (iv) if $n(a^2+b^2+c^2+\ldots)=(a+b+c+\ldots)^2$, then $a=b=c=\ldots$, n being the number of the letters.

35. Shew that the least value of $x+\dfrac{1}{x}$ is 2, the least value of $x+\dfrac{4}{x}$ is 4, and the least value of $x+\dfrac{9}{x}$ is 6, x being real and positive.

36. Shew that $\dfrac{x^2-3x+4}{x^2+3x+4}$ cannot be greater than 7 nor less than $\frac{1}{7}$, for real values of x.

37. If the sum of a given number of positive quantities is fixed, their continued product will be greatest when they are all equal.

38. If the continued product of a given number of positive quantities is fixed, their sum will be least when they are all equal.

39. Prove that, if $\dfrac{yz-x^2}{y+z}=\dfrac{zx-y^2}{z+x}$, then will each fraction be equal to $\dfrac{xy-z^2}{x+y}$, and also equal to $x+y+z$.

40. If $\dfrac{ad-bc}{a-b-c+d}=\dfrac{ac-bd}{a-b-d+c}$, then will $a=b$, or $c=d$, or $a+b=c+d$.

41. Shew that, if x, y, z be determined by the equations:

$$(a-\alpha)^2x+(a-\beta)^2y+(a-\gamma)^2z=(a-\delta)^2,$$
$$(b-\alpha)^2x+(b-\beta)^2y+(b-\gamma)^2z=(b-\delta)^2,$$
$$(c-\alpha)^2x+(c-\beta)^2y+(c-\gamma)^2z=(c-\delta)^2;$$

then will $\quad (d-\alpha)^2x+(d-\beta)^2y+(d-\gamma)^2z=(d-\delta)^2,$

where d has any value whatever.

42. If the equation $\dfrac{a}{x+a}+\dfrac{b}{x+b}=\dfrac{c}{x+c}+\dfrac{d}{x+d}$ have a pair of equal roots, then either one of the quantities a or b is equal to one of the quantities c or d, or else $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{1}{c}+\dfrac{1}{d}$. Prove also that the roots are then $-a$, $-a$, 0; $-b$, $-b$, 0; or 0, 0, $-\dfrac{2ab}{a+b}$.

43. If $2(x^2+x'^2-xx')(y^2+y'^2-yy')=x^2y^2+x'^2y'^2$, then will $x=x'$ and $y=y'$.

44. Shew that $a(a+d)(a+2d)(a+3d)+d^4$ is a perfect square.

45. Find the cube root of each of the following expressions:

(i) $x^3-24x^2y+192xy^2-512y^3$.

(ii) $1+3x+6x^2+7x^3+6x^4+3x^5+x^6$.

(iii) $x^6-9x^5+33x^4-63x^3+66x^2-36x+8$.

(iv) $8x^6-36x^5+102x^4-171x^3+204x^2-144x+64$.

46. Shew that

$$(y+z)(z+x)(x+y)+xyz=xyz(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).$$

47. If a, b, c, x are all real quantities, and

$$(a^2+b^2)x^2-2b(a+c)x+b^2+c^2=0,$$

then a, b, c are in geometrical progression and x is their common ratio.

48. Shew that $(1+x)(1+x^2)(1+x^4)\ldots$ to n factors $=\dfrac{1-x^{2^n}}{1-x}$.

49. If $s=a+b+c$, prove that

$$(as+bc)(bs+ca)(cs+ab)=(b+c)^2(c+a)^2(a+b)^2.$$

50. If $a+b+c+d=0$; then will

$$\begin{aligned}(a^3+b^3+c^3+d^3)^2&=9(bcd+cda+dab+abc)^2\\&=9(bc-ad)(ca-bd)(ab-cd)\\&=9(b+c)^2(c+a)^2(a+b)^2.\end{aligned}$$

CHAPTER XXIX.

SCALES OF NOTATION.

284. In arithmetic any number whatever is represented by one or more of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, called figures or digits, by means of the convention that every figure placed to the left of another represents *ten* times as much as if it were in the place of that other. The cipher 0, which stands for nothing, is necessary because one or more of the denominations units, tens, hundreds, &c., may be wanting.

The above mode of representing numbers is called the *common scale of notation*, and 10 is called the *radix* or *base*.

285. Instead of ten any other number might be used as the base of a *system of numeration*, that is of a system by which numbers are named according to some definite plan, and of the corresponding *scale of notation*, that is of a system by which numbers are represented by a few signs according to some definite plan.

Thus 4235 is said to be written in the scale of seven, if 3 stands for 3×7, 2 for $2 \times 7 \times 7$, and 4 for $4 \times 7 \times 7 \times 7$, so that every figure placed to the left of another represents *seven* times as much as if it were in the place of that other.

In general, any number N, is expressed in the scale of r, when it is written in the form $\dots d_3 d_2 d_1 d_0$, where each of the digits d_0, d_1, d_2, d_3... is zero or a positive integer less than r, and where d_0 stands for d_0 units, d_1 stands for $d_1 \times r$, d_2 stands for $d_2 \times r \times r$, and so on. Thus

$$N = d_0 + d_1 r + d_2 r^2 + \ldots.$$

286. *Any positive integer can be expressed in any scale of notation.*

Let N be the number and let r be the radix of the required scale.

Divide N by r, and let Q_1 be the quotient and d_0 the remainder.

Then $$N = d_0 + r \times Q_1.$$

Now divide Q_1 by r, and let Q_2 be the quotient and d_1 the remainder.

Then $$Q_1 = d_1 + r \times Q_2; \quad \therefore\ N = d_0 + rd_1 + r^2 Q_2.$$

By proceeding in this way we must sooner or later come to a quotient which is less than r; let this be after n divisions by r. The process is now complete and we have

$$N = d_0 + d_1 r + d_2 r^2 + \ldots + d_n r^n,$$

so that the number would in the scale of r be written

$$d_n \ldots d_3 d_2 d_1 d_0.$$

Each of the digits d_0, d_1, d_2,... is a positive integer less than r, and any one or more of them, except the last, d_n, may be zero.

Ex. 1. Express 1062 in the scale of 7.

The quotients and remainders of the successive divisions by 7 are as under:

$$\begin{array}{r|l} 7 & \underline{1062} \\ 7 & \underline{151} \quad \text{remainder } 5 = d_0 \\ 7 & \underline{21} \quad \ldots\ldots\ldots\ldots\ 4 = d_1 \\ & 3 \quad \ldots\ldots\ldots\ldots\ 0 = d_2 \end{array}$$

Thus 1062 when expressed in the scale of 7 is 3045.

Ex. 2. Change 2645 from the scale of 8 to the scale of 10.

$$\text{Since}\qquad 2645 = 2\times 8^3 + 6\times 8^2 + 4\times 8 + 5$$
$$= \{(2\times 8+6)\,8+4\}\,8+5\},$$

the required result may be obtained as follows :—*Multiply* 2 *by* 8 *and add* 6 ; *multiply this result by* 8 *and add* 4 ; *then multiply again by* 8 *and add* 5. This process is clearly applicable in all cases.

Ex. 3. Express 2156 in the scale of 5. *Ans.* 32111.

Ex. 4. Express 34239 in the scale of 11.

Ans. 237*t*7, where *t* is put for 10.

Ex. 5. Express $\frac{15}{17}$ as a fraction in the scale of 4. *Ans.* $\frac{33}{101}$.

Ex. 6. Change 31426 from the scale of 8 to the scale of 4.

Ans. 3030112.

We may first express 31426 in the scale of 10, as in Ex. 1, and reduce the result to the scale of 4, as in Ex. 2. The result may however be obtained by one process as under.

$$\begin{array}{r|l r}
4 & \underline{31426} & \\
4 & \underline{6305} & \text{remainder } 2 \\
4 & \underline{1461} & \ldots\ldots\ldots\ldots\ 1 \\
4 & \underline{314} & \ldots\ldots\ldots\ldots\ 1 \\
4 & \underline{63} & \ldots\ldots\ldots\ldots\ 0 \\
4 & \underline{14} & \ldots\ldots\ldots\ldots\ 3 \\
 & 3 & \ldots\ldots\ldots\ldots\ 0
\end{array}$$

Thus the number required is 3030112.

Explanation. We first divide 3 *eights* plus 1 by four, giving quotient 6 and remainder 1 ; we then divide 1 *eight* plus 4 by four, giving quotient 3 and remainder 0 ; and so on.

287. It would be a good exercise to perform all the ordinary rules of arithmetic with numbers expressed in various scales.

Ex. 1. Add 2345, 6127 and 1503. [Scale 8.]

Ex. 2. Subtract 3154 from 4021. [Scale 6.]

Ex. 3. Multiply 234 by 456. [Scale 7.]

Ex. 4. Divide 22326 by 315. [Scale 8.]

Answers 12177, 423, 150663, 56.

288. **Radix Fractions.** Radix fractions in any scale correspond to decimal fractions in the ordinary scale; thus $\cdot abc\ldots$, in the scale of r, stands for

$$\frac{a}{r}+\frac{b}{r^2}+\frac{c}{r^3}+\ldots$$

To shew that any given fraction may be expressed by a series of radix fractions in any proposed scale.

Let F be the given fraction; and suppose that, when expressed by radix fractions in the scale of r, we have

$$F=\cdot abc\ldots, \text{ that is } F=\frac{a}{r}+\frac{b}{r^2}+\frac{c}{r^3}+\ldots,$$

where each of $a, b, c, \ldots$ is zero or a positive integer less than r.

Multiply by r; then

$$F\times r=a+\frac{b}{r}+\frac{c}{r^2}+\ldots.$$

Hence a must be equal to the *integral* part, and $\frac{b}{r}+\frac{c}{r^2}+\ldots$ must be equal to the *fractional* part of $F\times r$. Let F_1 be the fractional part of Fr; then

$$F_1=\frac{b}{r}+\frac{c}{r^2}+\ldots.$$

Multiply by r again, then as before b must be equal to the integral part of $F_1\times r$.

Thus $a, b, c, \ldots$ can be found in succession.

Ex. 1. Express $\frac{55}{108}$ by a series of radix fractions in the scale of 6.

$$\frac{55}{108}\times 6=3+\frac{1}{18}\,;\ \frac{1}{18}\times 6=0+\frac{1}{3}\,;\ \frac{1}{3}\times 6=2.$$

Hence $\cdot 302$ is the required result.

Ex. 2. Change $431\cdot 45$ from the scale of 10 to the scale of 4.

The integral and fractional parts must be done by separate processes.

$$\begin{array}{r|l}
4 & \underline{431} \\
4 & \underline{107} \;\ldots\ldots\; 3 \\
4 & \underline{26} \;\ldots\ldots\; 3 \\
4 & \underline{6} \;\ldots\ldots\; 2 \\
 & 1 \;\ldots\ldots\; 2
\end{array}
\qquad\qquad
\begin{array}{r}
\cdot 45 \\
\underline{4} \\
1\cdot 80 \\
\underline{4} \\
3\cdot 2 \\
\underline{4} \\
0\cdot 8
\end{array}$$

Thus the required result is $12233 \cdot 1\dot{3}\dot{0}$.

289. **Theorem.** *Any number expressed in the scale of* r *is divisible by* r − 1 *if the sum of its digits is divisible by* r − 1.

Let N be the number, S the sum of the digits, and let the digits be d_0, d_1, d_2, etc.

Then $$N = d_0 + d_1 r + d_2 r^2 + \ldots + d_n r^n,$$

and $$S = d_0 + d_1 + d_2 + \ldots + d_n.$$

Hence $$N - S = d_1 (r - 1) + d_2 (r^2 - 1) + \ldots + d_n (r^n - 1).$$

Now each of the terms on the right is divisible by $r - 1$ [Art. 266].

Hence $N - S$ is divisible by $r - 1$, and therefore when S is divisible by $r - 1$ so also is N.

As a particular case of the above, any number expressed in the ordinary scale is divisible by 9 when the sum of its digits is divisible by 9.

290. **Theorem.** *Any number expressed in the scale of* r *is divisible by* r + 1 *when the difference between the sum of the odd digits and the sum of the even digits is divisible by* r + 1.

Let $$N = d_0 + d_1 r + d_2 r^2 + d_3 r^3 + \ldots,$$

and $$D = d_0 - d_1 + d_2 - d_3 + \ldots$$

Then $$N - D = d_1 (r + 1) + d_2 (r^2 - 1) + d_3 (r^3 + 1) + \ldots$$

Now each of the terms on the right is divisible by $r + 1$ [Art. 266].

Hence $N - D$ is always divisible by $r + 1$, and therefore when D is divisible by $r + 1$ so also is N.

As a particular case, any number expressed in the ordinary scale is divisible by 11 when the difference between the sum of the odd and the sum of the even digits is divisible by 11.

EXAMPLES. LXXVIII.

1. Express 2156, 7213 and 192457 in the scale of 6.

2. Change the following numbers from the scale of 7 to the ordinary scale: 2135, 4210, 30012, 123456.

3. Change 3152 and 23678 from the scale of 9 to the scale of 12.

4. Express 23·42 and 123·45 in the scale of 5.

5. Multiply $2{\cdot}3\dot{1}$ by $1{\cdot}2\dot{5}$, the numbers being expressed in the scale of 6.

6. Shew that the numbers represented in any scale by 121, 12321, and 1234321 are perfect squares.

7. Find the values of a and b in order that $215a14b3$ may be divisible by 9 and by 11.

8. Find the values of a and b in order that $516a7245b$ may be divisible by 99.

9. Find the scale in which 314 is represented by 626.

10. Find a number of two digits in the scale of 5 which is doubled by reversing its digits.

11. Find a number of two digits in the scale of 8 which is doubled by reversing its digits.

12. Find a number of two digits in the scale of 7 which is trebled by reversing the digits.

ANSWERS TO THE EXAMPLES.

I. **1.** 17. **2.** 6. **3.** 0. **4.** 168. **5.** $6\frac{2}{3}$.
6. $1\frac{5}{72}$. **7.** 1. **8.** 3. **9.** 1. **10.** 2. **11.** 3.
12. 2. **13.** 20. **14.** 4. **15.** 4. **16.** 50.
17. 140. **18.** 0. **19.** 15. **20.** $2\frac{2}{5}$. **21.** $\frac{5}{6}$.
22. $3, 4b, 5bc, 16abc$. **23.** $4, 5a, 7ab, 19abc$; 4, 5, 7, 19.

II. **1.** 16, 27, 64, 256, 8, 4, 2, 5, 5, 2. **2.** 13. **3.** 41.
4. 6. **5.** 145. **6.** 436. **7.** 240. **8.** 5.
9. $3\frac{4}{21}$. **10.** 79424. **11.** 608. **12.** 672. **13.** 75.
14. 11. **15.** 27. **16.** 36. **17.** 4140. **20.** 3.
21. 5. **22.** 17. **23.** 3. **24.** 4. **25.** $4\frac{1}{2}$.
26. 12. **27.** 36. **28.** 54. **29.** 3. **30.** 0.

III. **1.** 1. **2.** 1. **3.** 3. **4.** -1. **5.** -8.
6. -12. **7.** 11. **8.** 0. **9.** $2a-3b$. **10.** $-3a-2b$.
11. $5a-6b-2c$. **12.** $-3a-4b+7c$.

IV. **1.** $10a+3b+6c$. **2.** $2a-b-c$. **3.** a^2+3a+9.
4. $4a^3-3a^2-2a$. **5.** $-8a^2b+3ab^2-3b^3$. **6.** 0. **7.** $2a$.
8. $4x$. **9.** b. **10.** $2b$. **11.** a^3+a^2. **12.** $-3a+6a^3$.
13. $2m^2$. **14.** $pq-q^2$. **15.** $4a^2-4ab-\frac{1}{2}b^2$. **16.** 0.
17. 0. **18.** $2a^2+2ab+2b^2$. **19.** $a+b+c$. **20.** $2x$.
21. $3a-10b+2$. **22.** $-x^2y-2xy^2-3y^3$. **23.** 0. **24.** 0.
25. $\frac{5}{3}a^3-\frac{7}{15}a^2b-\frac{16}{15}ab^2+\frac{4}{3}b^3$. **26.** $5x^3-4x+15$.
27. $x^3+7ax^2-5a^3$. **28.** $-x^2-y^2$.

V. **1.** $2b$. **2.** $a+b$. **3.** $-2y$. **4.** $-\frac{1}{6}x-\frac{1}{6}y$.
5. $x+x^2$. **6.** $3-3x^2$. **7.** $-7a+6b-c$. **8.** $-b^2+6ab-7a^2$.
9. $-2x^2+10x-9$. **10.** $2x^2-2x$. **11.** $-\frac{4}{3}a+\frac{3}{2}b-\frac{1}{6}c$.
12. $-\frac{1}{4}y^2+\frac{1}{4}x^2$. **13.** $4b$. **14.** $-4a-4b$. **15.** $2ab$.
16. $-2x^2+xy$. **17.** $3x^3+x^2y-2xy^2-7y^3$. **18.** $-2x^3-6x^2y$.
19. $2a-3b$. **20.** a^2-ab-b^2. **21.** $2ab-ca+2bc$.
22. $-4a^2-2b^2-2c^2$. **23.** $-a$. **24.** $-3x^2-4x+9$.
25. $8a^2+8b^2+2c^2+6ab$.

VI. **1.** $2b$. **2.** $-2b$. **3.** $-2c$. **4.** z. **5.** $-y+z$.
6. 0. **7.** 3. **8.** 0. **9.** 1. **10.** 0. **11.** $6x-y-5z$.
12. $x-2y+5z+8$. **13.** $4a-4c$. **14.** $6x-4y+3z$.
15. $-2a-b+4c$. **16.** $2a-4b-3c+d$. **17.** $-9a$.
18. 0. **19.** $-8a^2+3ab-6b^2$. **20.** $5m^2-n^2+2mn$.

VII. **1.** $18a^2$. **2.** $35a^3$. **3.** $10a^5$. **4.** a^3b^4.
5. $6a^3b^3$. **6.** $28a^5b^5$. **7.** $18a^3b^2c^3$. **8.** $15a^2b^5c^2$.
9. $2a^4b^3c^2$. **10.** $-8ab$. **11.** $-12ab$. **12.** $-a^3$.
13. $-24a^4b^2$. **14.** $14a^7b^5$. **15.** $-18a^3b^3c^7$. **16.** $-6a^2b^5c^3$.
17. $10x^5y^7$. **18.** $-10a^3x^3y^7z^4$. **19.** $-72a^4b^5c^7x^7y^7z^6$.
20. $a^2, -a^3, a^4$. **21.** $x^4, -x^6, x^8$. **22.** $a^2b^2, -a^3b^3, a^4b^4$.
23. $a^4b^6, a^6b^9, a^8b^{12}$. **24.** $4a^6b^8, -8a^9b^{12}, 16a^{12}b^{16}$.
25. $9a^2b^4c^6, 4a^6b^2c^8, 16a^4b^6c^{10}$. **26.** $a^6, -a^{12}, a^3b^3, -a^6b^3$.
27. $8a^3b^6, -27a^9b^6, -64a^3b^{15}, -343a^6b^{15}c^{12}$.
28. $-a^2b^3, -8a^7, a^9b^9, a^{10}b^{18}$. **29.** -36. **30.** 24. **31.** 288.
32. -5. **33.** -20. **34.** 45. **35.** -63. **36.** 27.
37. 1. **38.** 25. **39.** 63. **40.** -3375.

VIII. **1.** $3a+3b$. **2.** $8a-4b$. **3.** $18a-24b$.
4. a^3+a^2. **5.** a^5-a^4. **6.** $3a^5+3a^3$. **7.** $4a^6-5a^5+a^4$.
8. $-6a^5+9a^4+12a^3$. **9.** $2a^4b^2-3a^3b^3+2a^2b^4$.
10. $ab^2c^2+a^2bc^2+a^2b^2c$. **11.** $-10x^5+15x^4-25x^3+20x^2$.
12. $-24x^3+18x^5-18x^6+24x^7$. **13.** $-15a^3b+10a^2b^2-35ab^3$.
14. $-12a^6b^4+18a^5b^5+30a^3b^6$. **15.** $6a+2b$. **16.** $2b-\frac{7}{4}c$.
17. $2bc$. **18.** $5ab-7ac+2bc$. **19.** $-a^2b^2d^2+a^2c^2d^2$.
20. $6ab-8ac+16ab$. **21.** $-9a$. **22.** $4a^3-2b^3+6bc^2$.
23. $33ac+12bc-12ab$.

IX. **1.** x^2-4y^2. **2.** a^2-9b^2. **3.** $6x^2+5xy-6y^2$.
4. $5a^2-ab-4b^2$. **5.** $x^2+13x+42$. **6.** $x^2-13x+42$.
7. x^2+x-42. **8.** $a^2+4a-45$. **9.** $4x^2+4x-24$.
10. $6x^2-17x+7$. **11.** $6y^2+7by-20b^2$. **12.** $9m^4-1$.
13. $4m^4-25n^4$. **14.** $a^2-\frac{1}{4}b^2$. **15.** $6a^2+2ab+\frac{1}{6}b^2$.
16. $\frac{1}{12}a^2-\frac{25}{144}ab+\frac{1}{12}b^2$. **17.** x^3-1. **18.** x^3+1.
19. a^3-b^3. **20.** a^3+b^3. **21.** $8a^3-27b^3$.
22. $64p^3-125q^3$. **23.** $x^4-7a^2x^2+6a^3x$.
24. $a^5-10a^3b^2+24a^2b^3$. **25.** $x^5-5x^3+x^2+7x+2$.
26. $x^5-x^3+5x^2-5x+2$. **27.** $x^4+x^2y^2+y^4$.
28. $a^8+a^4b^4+b^8$. **29.** $2x^5+3x^4y-3x^3y^2+x^2y^3+7xy^4+2y^5$.
30. $3x^5-16x^4y+39x^3y^2-53x^2y^3+42xy^4-15y^5$.

X. **1.** $a^2-2ab+b^2$. **2.** $-6a^3-4a^2+5a+2$.

3. $a^3+a^2b+ab^2+b^3$. **4.** $-6a^3+5a^2-2a-4$.

5. $a^3-3abc+b^3+c^3$.

6. $a^3+a^2b+a^2c+ab^2+ac^2+b^3+b^2c+bc^2+c^3$.

7. The degrees are 2nd, 3rd, 3rd, 3rd, 3rd, 3rd; also 1, 3, 5 and 6 are homogeneous.

8. $x^2+(a+b)x+ab$. **9.** $x^2-(a+b)x-ab$.

10. $x^3+(a+b+c)x^2+(bc+ca+ab)x+abc$.

11. $(a+b+c)x^2+(bc+ca+ab)x+abc$.

12. $x^2(y-z)-x(y^2-z^2)+yz(y-z)$.

13. $(a+b+c)x$. **14.** 0. **15.** $2bx+2cy$.

XI. **1.** $4a^2+b^2+4ab$. **2.** $16a^2+9b^2+24ab$.

3. $9a^2+b^2-6ab$. **4.** $25a^2+36b^2-60ab$.

5. $a^4+25a^2b^2-10a^3b$. **6.** $4a^4+9a^2b^2-12a^3b$.

7. $9x^2y^2+4y^4-12xy^3$. **8.** $16x^4+49y^4-56x^2y^2$.

9. $a^2+b^2+c^2-2ab-2ac+2bc$.

10. $4a^2+4b^2+c^2+8ab-4ac-4bc$.

11. $16a^2+4b^2+9c^2+16ab-24ac-12bc$.

12. $4a^2+25b^2+9c^2-20ab-12ac+30bc$.

13. $x^4+2x^3+3x^2+2x+1$. **14.** $x^4-2x^3+3x^2-2x+1$.

15. $x^4-2x^3y+3x^2y^2-2xy^3+y^4$.

16. $x^8+2x^6y^2+3x^4y^4+2x^2y^6+y^8$.

17. $a^2+b^2+c^2+d^2-2ab-2ac+2ad+2bc-2bd-2cd$.

18. $4a^2+4b^2+9c^2+9d^2-8ab-12ac+12ad+12bc-12bd-18cd$.

19. $9a^2+4b^2+16c^2+d^2+12ab-24ac+6ad-16bc+4bd-8cd$.

20. $25a^2+b^2+16c^2+9d^2+10ab-40ac-30ad-8bc-6bd+24cd$.

21. $x^6+2x^5+3x^4+4x^3+3x^2+2x+1$.

22. $x^6-2x^5+3x^4-4x^3+3x^2-2x+1$.

23. $4x^6-4x^5y+5x^4y^2-14x^3y^3+7x^2y^4-6xy^5+9y^6$.

24. $4x^6-4x^5y+9x^4y^2-8x^3y^3+6x^2y^4-4xy^5+y^6$.

25. $x^2+y^2-2xy-z^2$. **26.** $x^2+4y^2-4xy-16z^2$.

27. $9x^2+25z^2-30xz\quad y^2$. **28.** $4y^2+9z^2-12yz-x^2$.

29. $x^4+x^2y^2+y^4$. **30.** $9x^4+11x^2y^2+4y^4$.

31. $x^4-2x^3+x^2-49$. **32.** $4x^4+19x^2+49$.

33. $a^2+2ab+b^2-c^2-2cd-d^2$.

34. $4a^2-12ab+9b^2-4c^2+16cd-16d^2$.

35. $4a^2-8ad+4d^2-9b^2-6bc-c^2$.

36. $a^2-8ac+16c^2-9b^2+6bd-d^2$.

XII. 1. $3a^4-5a^3b-12a^2b^2-ab^3+3b^4$.
2. $3x^4-2x^3y-11x^2y^2-2xy^3+3y^4$.
3. $5x^5+16x^4y+17x^3y^2-12x^2y^3+18xy^4$.
4. $x^7-49x^5y^2+42x^4y^3-9x^3y^4$.
5. $6x^6+7x^5-54x^4+76x^3-74x^2+35x-12$.
6. $3a^6+8a^5-19a^4+36a^3-53a^2+58a-24$.
7. $\frac{3}{4}x^4-x^3y-\frac{11}{16}x^2y^2+xy^3-y^4$.
8. $3x^4-7\frac{5}{6}x^3y+143\frac{71}{144}x^2y^2+8\frac{1}{6}xy^3-3y^4$.
9. $a^3-3abc+b^3+c^3$. 10. $8a^3-18abc+27b^3+c^3$.
11. $a^3+3abc+b^3-c^3$. 12. $27a^3+27abc-b^3+27c^3$.
13. $x^4+x^2y^2+y^4$. 14. $a^4+4a^2b^2+16b^4$.
15. $a^5x^5+a^3x^3+ax$. 16. x^4-1. 17. a^4-x^4.
18. x^4-16y^4. 19. $81x^4-625y^4$. 20. x^8-y^8.
21. $a^8-2a^4b^4+b^8$. 22. $x^8+2x^6+3x^4+2x^2+1$.
23. $x^8+x^4y^4+y^8$. 24. $a^8+a^4b^4+b^8$. 25. a^6-x^6.
26. $2y^2z^2+2z^2x^2+2x^2y^2-x^4-y^4-z^4$.
27. $8a^3+36a^2b+54ab^2+27b^3$.
28. $8a^3-36a^2b+54ab^2-27b^3$. 29. $27a^3-54a^2b+36ab^2-8b^3$.
30. $a^3+b^3+c^3+3\,(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)+6abc$.
31. $a^3-b^3+c^3+3\,(-a^2b+ab^2+b^2c-bc^2+c^2a+ca^2)-6abc$.
32. $a^3-b^3-c^3+3\,(-a^2b+ab^2-b^2c-bc^2+c^2a-ca^2)+6abc$.
37. $8xz$. 38. $x^3+y^3+z^3$.

XIII. 1. -2. 2. -5. 3. $\frac{2}{3}$. 4. $-a$. 5. $-4a$.
6. $-\frac{4}{3}bc$. 7. $-12ab$. 8. $\frac{3}{2}xy^2$. 9. $-\frac{3}{2}x^3y$.
10. $-5ab^5$. 11. $\frac{9}{2}b^8c^4$. 12. $-cd$. 13. $-\frac{1}{2}abc^3d^2$.
14. $\frac{4}{3}a^3x^2y^3z$. 15. $\frac{3}{2}ab^5x^4y^4$. 16. $3x-5a$.
17. $-5y^2+6y$. 18. $4a^2-5a+2$. 19. $-4a-3a^2+2a^3$.
20. $-5a^2b^3+\frac{7}{3}ab^4-3b$.

XIV. 1. $x-3$. 2. $x-3$. 3. $x-12$. 4. $x-11$.
5. $3x+2$. 6. $x-2$. 7. $a+b$. 8. $x+3y$.
9. $x-4y$. 10. $3x-8y^2$. 11. $x+\frac{1}{2}y^2$.
12. $\frac{1}{3}x^3+\frac{1}{4}a^3$. 13. $\frac{1}{2}x+\frac{1}{3}$. 14. $\frac{1}{4}x-\frac{1}{3}$.
15. a^2+ab+b^2. 16. $4a^2-6ab+9b^2$.
17. $a^2x^2-abxy+b^2y^2$. 18. $4a^4x^4+10a^2x^2y^2+25y^4$.
19. $-x^2-3x-1$. 20. $-2x^2+x+2$.
21. $-2x^3+3x^2-x+2$. 22. x^2+x+2.

XV. 1. x^2-x+1. 2. x^4+x^2+1. 3. x^2-2x+4.
4. $4x^2+6x+9$. 5. x^2-x+2. 6. $2x^2+x+1$.
7. $-x^4-2x^3-2$. 8. $-x^5-2x^4-4x^3-8x^2-13x-26$.
9. $-x^2+x+1$. 10. x^4-3x^2+4x+1. 11. $3x^2+2x+1$.
12. $4x^3+3x^2+2x+1$. 13. $2x^2-3x+5$.
14. $-3x^2+3x+1$. 15. x^3+2x^2+3x-1.
16. $4x^4-11x^3+5x^2-11x+8$. 17. $2x^5-x^4+x^3-x+1$.
18. $x^8-x^7+x^5-x^4+x^3-x+1$. 19. $3x^3+x^2y-4xy^2+2y^3$.
20. $2y^3-4xy^2+x^2y+3x^3$. 21. $x+y-z$. 22. $x+y+2z$.
23. $a-b$. 24. $-x-y+z$. 25. $-a-2b-c$.
26. $-a-2b+3c$. 27. $2a-b-c$. 28. $3a-4b+c$.
29. $x^2-xy+xz+y^2+yz+z^2$.
30. $-4x^2-2xy+2xz-y^2-yz-z^2$.
31. $9a^2+6ab-3ac+4b^2+2bc+c^2$.
32. $a^2+2ab-ac+b^2-bc+c^2$.
33. $a^3-a^2b-a^2c-ab^2+2abc-ac^2+b^3-b^2c-bc^2+c^3$.
34. $(x+1)^2-y(x+1)+y^2$. 35. $x^4-4x^2yz+7y^2z^2$.
36. $x^4-x^2yz+7y^2z^2$. 37. $3a+b+2c+d$.
38. $x^2+(a-2b)x+a^2+3b^2$. 39. $a+b+c$.
40. $a^2+a(b+c)+b^2+bc+c^2$.

MISCELLANEOUS EXAMPLES. I.

A. 1. 6. 2. $3a^2+ab-4ac+2b^2+bc+c^2$.
4. $x^3+\frac{23}{4}x^2-\frac{4}{3}x+1$, $\frac{1}{16}x^4+\frac{1}{4}x^2y^2+y^4$. 5. $12x^2+12$.
6. $2+4x+6x^2+8x^3+10x^4$.

B. 1. 87, -5. 2. $5b^4-3b^3a+3ba^3-5a^4$.
3. 0, $7x-7$, $2b-2a$. 4. $9x^4-\frac{1}{9}$, $a^2+2ab+b^2-c^2$.
6. $x^2-(y+2)x+y^2+y+1$.

C. 1. $-1\frac{1}{6}$. 2. $2a^3+\frac{5}{2}a^2b+\frac{31}{5}b^3$.
3. $a^2x^4+(2a^2-1)x^2+a^2$, $4a^2+b^2+9c^2+4ab-12ac-6bc$.
4. $x^2+2xy+y^2$. 5. $x^4-xy^3+y^4$.

D. 1. 0, 0. 2. $2x-12y+7z$. 3. $-4ab-4ac$.
4. $a^3-2a^2b+2ab^2-b^3$. 5. x^2-xz+z^2, $(x+y)^2-(x+y)z+z^2$.

E. 1. 8. 2. $14x+2y$, $3x+15y$, $42x^2+216xy+30y^2$.
4. b^2-a^2. 5. $5(b-a)$, $5(7a^2-11ab+7b^2)$.

F. **1.** $\frac{1}{4}$. **2.** $a^3 - a^2(x + \frac{1}{2}y + \frac{1}{4}z) + a(\frac{1}{2}xy + \frac{1}{4}xz + \frac{1}{8}yz) - \frac{1}{8}xyz$.
3. 12. **4.** $x - y + z$, $2y^2$.

G. **1.** $41x - 51y$. **2.** $-8a^2 + 6ab - 10b^2$. **3.** $a^3 - 3ab + b^3 + 1$.
4. $a^2 + 2ab + 2ac + b^2 + 2bc + 4c^2$. **5.** $x^5 + 2x^4 - 8x^2 - 16x$.

XVI. **1.** 3. **2.** 1. **3.** -4. **4.** -6. **5.** 0.
6. 0. **7.** $\frac{3}{4}$. **8.** $\frac{1}{13}$. **9.** -2. **10.** -3.
11. $5\frac{4}{9}$. **12.** 2. **13.** -8. **14.** 6. **15.** -20.
16. 5. **17.** -5. **18.** $-2\frac{12}{25}$. **19.** 12. **20.** $-9\frac{5}{13}$.
21. $2\frac{1}{2}$. **22.** 13. **23.** 1. **24.** $-1\frac{3}{4}$. **25.** $\frac{2}{3}$.
26. $2\frac{1}{2}$. **27.** $-1\frac{1}{4}$. **28.** $-1\frac{1}{6}$. **29.** $4\frac{1}{2}$. **30.** $-1\frac{3}{19}$.
31. -5. **32.** -1. **33.** 2. **34.** -16. **35.** 1.
36. 1. **37.** 40. **38.** 2. **39.** 20. **40.** $a + b$.
41. $2a$. **42.** $b - a$. **43.** $\frac{a}{2}$. **44.** $\frac{b}{2}$. **45.** a.
46. ab. **47.** $-\frac{2ab}{a^2 + b^2}$. **48.** 0. **49.** $b - a$.
50. $-\frac{a^2 + b^2}{2a}$. **51.** $\frac{1}{2}(a + b)$. **52.** $\frac{bc}{a}$. **53.** $-\frac{1}{2}(a + b)$.
54. $\frac{1}{3}(a + b + c)$.

XVII. **1.** 99, 101. **2.** 18, 38. **3.** 10, 15. **4.** $27\frac{1}{2}$, $72\frac{1}{2}$.
5. 20. **6.** 21. **7.** 20, 5. **8.** 15, 5.
9. 12, 26. **10.** 9, 22. **11.** 40. **12.** 420.
13. 14. **14.** 6. **15.** 420. **16.** 112.
17. 45, 55. **18.** 60, 40. **19.** $13\frac{1}{3}$, $33\frac{1}{3}$. **20.** 24, 12.
21. £20. **22.** 25*s*., 15*s*. **23.** £18. 15*s*.
24. £150, £100. **25.** £40, £35. **26.** £130, £125, £105.
27. £30, £15, £20. **28.** *A* £22. 10*s*., *B* £27. 10*s*., *C* £50.
29. £1 to each man, 10*s*. to each woman, 5*s*. to each child.
30. 1*s*. 8*d*. to each man, 8*d*. to each woman, 4*d*. to each child.
31. In 5 years. **32.** 30 years ago.
33. 30, 10. **34.** 35, 5. **35.** *A* £25, *B* £35, *C* £40.
36. *A* £$23\frac{1}{2}$, *B* £$25\frac{1}{2}$, *C* £$27\frac{1}{2}$, *D* £$23\frac{1}{2}$. **37.** 10*d*. a lb.
38. 10*d*. a score. **39.** £12000. **40.** £1000, £500, £250.
41. 20 shillings, 4 half-sovereigns, and 4 sovereigns.
42. 4 sovereigns, 12 half-sovereigns, and 20 shillings.
43. 5 h. $27\frac{3}{11}$ min. **44.** 9 h. $32\frac{8}{11}$ min. **45.** 18 and 15.
46. 85 s., 35 w. **47.** 1500. **48.** 28. **49.** £270.
50. 30. **51.** £280. **52.** 3*s*. **53.** 15 days.
54. 8 days. **55.** 480.

XVIII. **1.** $1, -\frac{3}{2}$. **2.** 3, 2. **3.** 100, 9. **4.** $-2, -1$.

5. 1, 2. **6.** $2, -1$. **7.** 5, 3. **8.** 11, 8.

9. $\frac{11}{19}, \frac{3}{19}$. **10.** $4, -3$. **11.** $1, -1$. **12.** $-\frac{435}{8}, \frac{11}{8}$.

13. $\frac{5}{6}, -\frac{5}{6}$. **14.** $-\frac{62}{67}, \frac{55}{134}$. **15.** 1, 1. **16.** $5, -3$.

17. $\frac{23}{7}, \frac{47}{7}$. **18.** 5, 2. **19.** $5, -3$.

XIX. **1.** 1, 3. **2.** $5, -3$. **3.** 7, 5. **4.** $4, -1$.

5. $\frac{1}{5}, \frac{1}{2}$. **6.** $\frac{1}{2}, \frac{5}{2}$. **7.** $1, -1$. **8.** $3, -1$.

9. 5, 2. **10.** $\frac{207}{224}, \frac{425}{224}$. **11.** 8, 16. **12.** 5, 7.

13. $-2, -2$. **14.** $\frac{5}{2}, \frac{3}{2}$. **15.** $\frac{18}{7}, \frac{8}{3}$. **16.** $\frac{19}{69}, -19$.

17. 3, 2. **18.** $\frac{23}{7}, -\frac{23}{33}$. **19.** 3, 6. **20.** $\frac{1}{2}, -\frac{3}{2}$.

21. $\frac{24}{23}, -\frac{41}{24}$. **22.** $\frac{1}{3}, -4$. **23.** $1, -\frac{1}{4}$. **24.** $a+b,\ a-b$.

25. $a+b,\ a+b$. **26.** $a,\ b$. **27.** $2a+b,\ a+2b$.

28. $a-b,\ a-b$. **29.** $\frac{a}{b}, \frac{b}{a}$. **30.** $2b-a,\ 2a-b$.

31. $a^2+b^2,\ ab$. **32.** $a+b,\ a-b$. **33.** $\frac{c^2(c-b)}{a^2(a-b)}, \frac{c^2(c-a)}{b^2(b-a)}$.

34. $\frac{1}{a+b}, \frac{1}{a+b}$. **35.** $a+b,\ a+b$. **36.** 1, 0.

37. $a+\frac{b}{2}, a-\frac{b}{2}$.

XX. **1.** 14, 10, 4. **2.** $b+c-a,\ c+a-b,\ a+b-c$.

3. $-3, 3, 1$. **4.** $-1, 0, 1$. **5.** $-1, -2, 4$. **6.** $\frac{61}{7}, \frac{22}{7}, 3$.

7. The equations are not independent, for *any* values which satisfy two of the equations will satisfy the third.

8. 3, 1, 2. **9.** The equations are inconsistent.

10. $-\frac{1}{2}, \frac{19}{15}, \frac{7}{30}$. **11.** $\frac{1}{2a}, \frac{1}{2b}, \frac{1}{2c}$. **12.** $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$.

XXI. **1.** £20, £30. **3.** £12½, £2½. **4.** $\frac{2}{3}$ tons, $\frac{3}{2}$ tons.

5. *A*, 40 days, *B*, 120 days. **6.** $\frac{3}{5}$. **7.** $\frac{2}{3}$.

8. 240, 360. **9.** £22, £26. **10.** 5 shillings.

11. 480 yards. **12.** 24 days. **13.** £2400, £900.

15. 12, 24, 36, 48. **16.** *A*, £450, *B*, £225, *C*, £237½, *D*, £87½.

17. 48. **18.** 7 and 2. **19.** $\frac{31}{43}$. **20.** 1*s*. 6*d*.

21. 200, 250. **22.** 3 half-crowns, 8 shillings, 9 sixpences.

23. 17½ francs. **24.** 3*s*. 6*d*., 4*s*. 2*d*. **25.** 50.

MISCELLANEOUS EXAMPLES. II.

A. 1. 1. 2. $7a-7b$. 4. x^2-2x-2.
5. (i) 2, (ii) $x=6$, $y=15$, (iii) $x=y=a+b$.
6. 60.
B. 1. $4\frac{1}{4}$. 2. a^2+b^2. 4. x^4-2x^2+x+2.
5. (i) -6; (ii) $x=-162$, $y=42$; (iii) $x=\frac{54}{13}$, $y=\frac{216}{41}$.
6. 90 sovereigns, 40 crowns.
C. 1. $-b-d$.
4. $a^2+b^2+c^2-bc-ca-ab$, $4x^2+4y^2+z^2-2yz-2zx-4xy$.
5. (i) $x=-1$, $y=-1$; (ii) $x=a^2+b^2$, $y=-ab$.
6. A, 12 hits and 18 misses. B, 24 hits and 6 misses.
D. 1. $-11\frac{1}{4}$. 2. $2ab$. 4. $15b^2c^3-3ab^3c$, $a+2b-3c$.
5. (i) $\frac{7}{5}$; (ii) $x=5$, $y=4$; (iii) $x=\frac{1}{4}$, $y=\frac{1}{2}$. 6. 1225.
E. 1. a^2. 2. $4a^2+ab+ac+4b^2+bc+4c^2$.
4. $x^2-\frac{1}{2}x+\frac{2}{3}$, cx^2+dx-c.
5. (i) 5; (ii) $x=\frac{3}{10}$, $y=\frac{3}{2}$; (iii) $x=a+b$, $y=a-b$.
6. 8 and 5.
F. 1. 130. 2. a^8-x^8. 4. $2y^2-3y+1$, $a+2b+3c$.
5. (i) $\frac{cd-ab}{a+b-c-d}$; (ii) $x=-1$, $y=-1$.
6. Tea 2*s.* 6*d.*, coffee 1*s.* 8*d.*

XXII. 1. $x(x+1)$. 2. $a(a-b)$. 3. $b(a-c)$.
4. $x(2a+\frac{1}{2}x)$. 5. $x^2(4x-3)$. 6. $a^2(a-3b)$.
7. $x^2(x^2-5xy+20y^2)$. 8. $a^2(a^4-a^2x+x^2)$.
9. $ax^2(6x^2-5a^2+20ax)$. 10. $a^2x^3y^2(3ax-\frac{1}{9}y)$.
11. $abc^2(11a-\frac{1}{5}bc)$. 12. $p^2q^3(r^6-7p^4q^2)$.

XXIII. 1. $(2x+1)^2$. 2. $(3x-1)^2$. 3. $(1-4x^2)^2$.
4. $(2a-3b)^2$. 5. $(3a^2+4b^2)^2$. 6. $(x+\frac{1}{2}y)^2$.
7. $(2ax+by)^2$. 8. $(5a^2x-3b^2y)^2$. 9. $3(a+b)^2$.
10. $5(a^2-b)^2$. 11. $a(a-3b)^2$. 12. $3a^3(a-5b^3)^2$.
13. $-(x^2-2y^2)^2$. 14. $-(2x^2-2)^2$. 15. $xy(2y-x)^2$.
16. $xy(x+\frac{1}{2}y)^2$. 17. $(a+b+2c)^2$. 18. $(x^2+y^2-z^2)^2$.
19. $(2xy+a+b)^2$. 20. $\{3(a+b)-c\}^2$.

XXIV. 1. $(a-3)(a+3)$. 2. $(4-b)(4+b)$. 3. $(5a-b)(5a+b)$.
4. $(x-3y)(x+3y)$. 5. $(4x-3y)(4x+3y)$.
6. $(8a-7b)(8a+7b)$. 7. $(2a-9b)(2a+9b)$.

8. $(x-3y^2)(x+3y^2)$. **9.** $(6a^2-7b^2)(6a^2+7b^2)$.

10. $(2ab-3c)(2ab+3c)$. **11.** $(3ax-7by)(3ax+7by)$.

12. $(7abc-6xyz)(7abc+6xyz)$. **13.** $x(2y-3x)(2y+3x)$.

14. $2a(2b-3a)(2b+3a)$. **15.** $3a^3(a-6)(a+6)$.

16. $7a^5(1-2a^2)(1+2a^2)$. **17.** $8xy(x-2y)(x+2y)$.

18. $7ab(c-ab)(c+ab)$. **19.** $(x^2+y^2)(x+y)(x-y)$.

20. $(x^2+4y^2)(x+2y)(x-2y)$.

21. $(9a^2+4b^2)(3a+2b)(3a-2b)$.

22. $(25a^2+16x^2)(5a+4x)(5a-4x)$.

23. $(x^2y^2+a^2b^2)(xy+ab)(xy-ab)$.

24. $(a^2b^2+9c^2d^2)(ab+3cd)(ab-3cd)$.

25. $(9x^2y^2+1)(3xy+1)(3xy-1)$.

26. $(4a^2b^2c^2+1)(2abc+1)(2abc-1)$.

27. $(4+9x^2y^2)(2+3xy)(2-3xy)$.

28. $(x^4+y^4)(x^2+y^2)(x+y)(x-y)$.

29. $(a^4+b^4c^4)(a^2+b^2c^2)(a+bc)(a-bc)$.

30. $a^2(a^4+1)(a^2+1)(a+1)(a-1)$. **31.** $(a+b+c)(a+b-c)$.

32. $(a+b+2c)(a+b-2c)$. **33.** $(2x+2y+1)(2x+2y-1)$.

34. $\{3(x-y)+2\}\{3(x-y)-2\}$. **35.** $4xy$.

36. $3(a+b)(a-b)$. **37.** $y(2x-y)$. **38.** $4b(a-b)$.

39. $(3a+b)(a+3b)$. **40.** $(5x+y)(x+5y)$.

41. $(a+b)^2(a-b)^2$. **42.** $4a(b+c)$.

43. $(4a+4b-3c)(2a-2b-c)$. **44.** $4(a-b+c)(2c-b)$.

45. $8x(x+1)^2(x-1)$.

XXV. **1.** $(a-2b)(a^2+2ab+4b^2)$. **2.** $(2a+b)(4a^2-2ab+b^2)$.

3. $(2a-5x)(4a^2+10ax+25x^2)$. **4.** $(a-5x^2)(a^2+5ax^2+25x^4)$.

5. $4(a+2b)(a^2-2ab+4b^2)$. **6.** $(3x-\frac{1}{2}y)(9x^2+\frac{3}{2}xy+\frac{1}{4}y^2)$.

7. $(xy+\frac{1}{3}ab)(x^2y^2-\frac{1}{3}abxy+\frac{1}{9}a^2b^2)$.

8. $(2a^2b^2+x^2)(4a^4b^4-2a^2b^2x^2+x^4)$.

9. $2x(y-\frac{1}{2}x)(y^2+\frac{1}{2}xy+\frac{1}{4}x^2)$. **10.** $9ab^2(a-\frac{1}{3}b)(a^2+\frac{1}{3}ab+\frac{1}{9}b^2)$.

11. $3ab(a+2b)(a^2-2ab+4b^2)$.

12. $5bc(2a-bc)(4a^2+2abc+b^2c^2)$.

13. $(a^3+8)(a^3-8)=(a+2)(a-2)(a^2+2a+4)(a^2-2a+4)$.

14. $(8a^3+27b^3)(8a^3-27b^3)=(2a+3b)(2a-3b)(4a^2+6ab+9b^2)$
$(4a^2-6ab+9b^2)$.

15. $(x^6-a^3b^3)(x^6+a^3b^3)=(x^2-ab)(x^2+ab)(x^4-abx^2+a^2b^2)$
$(x^4+abx^2+a^2b^2)$.

16. $(x+y)(x^2+5xy+7y^2)$.

17. $3(x+y)\{(x+2y)^2-(x+2y)(2x+y)+(2x+y)^2\}$
$=9(x+y)(x^2+xy+y^2)$.

18. $3(y-x)\{(2y-x)^2+(2y-x)(2x-y)+(2x-y)^2\}$
$=9(y-x)(x^2-xy+y^2)$.

19. $4(x-y)\{(x-3y)^2+(x-3y)(y-3x)+(y-3x)^2\}$
$=4(x-y)(7x^2-2xy+7y^2)$.

20. $(x+y)\{(2y-x)^2-(2y-x)(2x-y)+(2x-y)^2\}$
$=(x+y)(7x^2-13xy+7y^2)$.

XXVI. 1. $(x+1)(x+3)$. 2. $(x-1)(x-3)$. 3. $(x-2)(x-4)$.
4. $(x-3)(x-5)$. 5. $(x-2)(x-9)$. 6. $(x+4)(x+5)$.
7. $(x+3)(x-1)$. 8. $(x+5)(x-1)$. 9. $(x+3)(x-2)$.
10. $(x-3)(x+2)$. 11. $(x+7)(x-5)$. 12. $(x-5)(x+2)$.
13. $(x+7)(x-2)$. 14. $(x-12)(x+11)$. 15. $(x+12)(x+6)$.
16. $(x-12)(x+7)$. 17. $(x-10)(x-15)$. 18. $(x+15)(x-10)$.
19. $(x+20)(x-9)$. 20. $(x+12)(x-13)$. 21. $(x-15)(x-16)$.
22. $(x-25)(x+8)$. 23. $(x-16)(x-18)$. 24. $(x-40)(x+5)$.

XXVII. 1. $(3x-1)(x-3)$. 2. $(3x-2)(x-5)$. 3. $(2x+3)(x+4)$.
4. $(2x-1)(x+2)$. 5. $(3x-2)(x+3)$. 6. $(4x-3)(x+1)$.
7. $(5x-3)(x-7)$. 8. $(3x-4)(x+5)$. 9. $(7x+9)(x-6)$.
10. $(5x-8)(x-6)$. 11. $(7x-9)(x+12)$. 12. $(9x-5)(x+15)$.
13. $(4x-3)(x+6)$. 14. $(2x+5)(2x-3)$. 15. $(6x-5)(x+10)$.
16. $(5x-1)(2x+1)$. 17. $(12x-1)(11x+1)$.
18. $(x-1)(4x-1)$. 19. $2(6x-5)(x+5)$.
20. $(7x-3)(x+18)$. 21. $3(4x+5)(2x-5)$.
22. $(x+y)(x+3y)$. 23. $(x-2y)(x-4y)$.
24. $(x-2y)(x-9y)$. 25. $(x+7y)(x-2y)$.
26. $(x-10y)(x-15y)$. 27. $(x+5y)(x-40y)$.
28. $(3x-2y)(x-5y)$. 29. $(7x+9y)(x-6y)$.
30. $(6x+5y)(4x-15y)$.
31. $(x^2-4)(x^2-9)=(x-2)(x+2)(x-3)(x+3)$.
32. $(x^2-16y^2)(x^2-9y^2)=(x-4y)(x+4y)(x-3y)(x+3y)$.
33. $(9x^2-4y^2)(4x^2-9y^2)=(3x-2y)(3x+2y)(2x-3y)(2x+3y)$.
34. $x(x-6)(x+3)$. 35. $xy(x+y)(x-2y)$.
36. $x^2y(5x+2y)(3x-2y)$. 37. $xy^3(15+xy)(5-9xy)$.

XXVIII. 1. $(2x-5y)(2x+5y)(4x^2+25y^2)$.
2. $(3a-2b)(3a+2b)(9a^2+4b^2)$. 3. $x(x-3)(x+3)(x^2+9)$.
4. $(1+3a)(1-3a+9a^2)$. 5. $(3+2x)(9-6x+9x^2)$.

6. $x(3x+2)(9x^2-6x+4)$. 7. $(a+b+c-d)(a+b-c+d)$.
8. $(a+b+2c-2d)(a+b-2c+2d)$. 9. $(b-c)(2a+b+c)$.
10. $8xy(x^2+y^2)$. 11. $(a+b+3c)(a+b-c)$.
12. $(a+b)(a+b-6c)$. 13. $ab(a+2b)(2a+b)$.
14. $4ab(a^2+b^2)$. 15. $(x+1)^3(x-1)^3$. 16. $12xy(x+y)^2$.
17. $4(a+c)(b+d)$. 18. $8(a-c)(a+b+c)$.
19. $5(x+y)(7x^2+11xy+7y^2)$. 20. $2(b-a)(13a^2+22ab+13b^2)$.
21. $(x-y)(x+\frac{1}{4}y)$. 22. $(x+ay)(x+\frac{1}{a}y)$. 23. $x(x+\frac{2}{3})(x+\frac{3}{2})$.
24. $(3x+2y)(2x-3y)$. 25. $x^2(x+\frac{2}{3}y)(x-\frac{3}{2}y)$.
26. $(9x+8)(8x-9)$. 27. $(x-1)(x+1)(x-2)(x+2)$.
28. $(x-2y)(x+2y)(x-3y)(x+3y)$. 29. $(ax+by)(bx+ay)$.
30. $(ax-by)(bx+ay)$. 31. $(a+b-2c)(a+b-3c)$.
32. $(x+y-5z)(x+y-2z)$. 33. $\{a+b-3(c+d)\}\{a+b-5(c+d)\}$.
34. $(x-y)(x+y+2)$. 35. $(x-y)(x+y+4)$. 36. $(x^2+1)(x-5)$.
37. $(x+1)(x-2)(x+2)$. 38. $(x-1)(x+1)(2x-3)$.
39. $(5x-1)(x-1)(x+1)$. 40. $(x+1)\{a(x^2-x+1)+1\}$.
41. $(x+1)\{a(x^2-x+1)+b\}$. 42. $(x+b)(x-a)(x+a)$.
43. $(bx+a)(x^2+1)$. 44. $(x+y)(ax+by)$. 45. $(a^2+1)(b^2+1)$.
46. $(a-1)(a+1)(b-1)(b+1)$. 47. $(a-b)(c-d)$.
48. $(a-b)(c-d)(c+d)$. 49. $(x-y)(x+y+z)$. 50. $2b(a-b)$.
51. $(a-c)(a+c)(a^2+b^2+c^2)$. 52. $(a-b)(a+b-c)$.
53. $(a-c)(a+c-1)$. 54. $(1-ax)(1+ax+bx)$.
55. $(1-ax)(1+ax+bx^2)$. 56. $(ac+b)(ac+d)$.
57. $(a+b)(ax+by+c)$. 58. $(a-c+b-d)(a-c-b+d)$.
59. $(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$.
60. $(a+b+c+d)(a+b-c-d)(a+c-b-d)(a-b-c+d)$.
61. $(x^2+5x+1)(x^2-5x+1)$. 62. $(x^2+3xy+y^2)(x^2-3xy+y^2)$.
63. $(x^2+3xy-y^2)(x^2-3xy-y^2)$. 64. $(x^2+x-1)(x^2-x-1)$.
65. $(x^2+4x+3)(x^2+4x-5)=(x+1)(x+3)(x^2+4x-5)$.
66. $(x^2+7x+26)(x^2+7x-8)=(x^2+7x+26)(x-1)(x+8)$.

XXIX. 1. a^2b^2. 2. abc^2. 3. $3ab$. 4. $2xy$.
5. $12a^2b^3x^4$. 6. b^3x. 7. x^3y^5. 8. b^3c^6.
9. $7y^2$. 10. ab. 11. xyz^2. 12. $abcx^2$.

XXX. 1. $x^2(x-a)^3$. 2. $a+b$. 3. $(a+b)^2$.
4. $b^4c^3(b+c)^2$. 5. $a(a^2+b^2)$. 6. a^2+3b^2.
7. $a^2x^2(x+2a)$. 8. $a^2x^2(a^2-4x^2)$. 9. $x+2$.
10. $x(x+2y)$. 11. $x-1$. 12. $a-b$.

XXXI. 1. $x-1$. 2. $x-y$. 3. $2x-1$. 4. $2x-y$.
5. x^2-2. 6. x^2-2y. 7. $a-2$. 8. $2b-1$.
9. $x-1$. 10. x^2y^2-1. 11. $x-2a$. 12. $2a-b$.
13. a^2-ab+b^2. 14. $4a^2-2a+1$. 15. x^2-y^2.
16. $x-1$. 17. x^2+x+1. 18. x^2-3x+5.
19. $x-7$. 20. $x-y$. 21. $x-y$. 22. $x-5$.
23. $4x^2-2x+1$. 24. $a-2x+3$. 25. x^2+7.
26. x^2+5x+1. 27. x^2+4x+3. 28. x^2+3x+1.
29. $y-2$. 30. $y^2-3yz+4z^2$. 31. $2x^2-3x-1$.
32. x^2+2x+2. 33. $x-1$. 34. x^2-3x+1.
35. $x^2+(2m-3)x-6m$. 36. $mx+ny$.

XXXII. 1. a^3b^3. 2. a^2bc^3. 3. $18a^2b^3$. 4. $20x^3y^3$.
5. $120a^3b^4x^6$. 6. $3a^2b^5x^5$. 7. $72a^2b^3x^4y^6$. 8. $6a^3b^3c^8$.
9. $462ab^3xy^4z^3$. 10. $a^2b^3c^2$. 11. $30x^2y^3z^3$. 12. $a^4b^2c^3x^4$.

XXXIII. 1. $(a-x)(a-2x)(a-3x)$.
2. $a^3x^3(a-x)(a-2x)(a-3x)$. 3. $(a-b)(a+b)^2$.
4. $12a^2b(a-b)(a+b)^2$. 5. $(x+1)(x+2)(x+4)$.
6. $(x+y)(x+2y)(x+4y)$. 7. $(x-1)(x-2)(x-3)$.
8. $(3x-y^2)(2x-y^2)(x-y^2)$. 9. $(a^2-b^2)^2$. 10. $(x^2-4y^2)^2$.
11. $(x+2)(x+3)(x+4)$. 12. $(x-2y)(x-3y)(x-4y)$.

XXXIV. 1. $54a^3b^2c^2$. 2. $a^4b^3c^3$. 3. $24a^3b^2$. 4. $12a^3b^3$.
5. $x^3y^3(x^2-y^2)^2$. 6. $a^2(a^2-b^2)$. 7. $xy^2(x^2-1)$.
8. $12ax^2y^2(x-y)(x+y)^2$. 9. $90(x-3)(x+3)(x-4)$.
10. $12(a^2-b^2)(a^2-4b^2)$. 11. $8ab^3d(a^2-d^2)$.
12. $(x-2)(x-4)(x-6)$. 13. $(x+1)(x+3)(x-10)$.
14. $6(x^2-1)(x^2-4)$. 15. $(x-1)(x-2)(2x+3)(3x+1)$.
16. $(2x-1)(3x+2)(7x+1)$. 17. $(x-3y)(x-y)(3x-y)$.
18. $x(x-1)(x+3)(2x-1)$. 19. $xy^2(x^2-4)(x^2-9)$.
20. x^6-a^6. 21. x^4-1. 22. $(x-4)(3x-2)(3x^2+2x+1)$.
23. $(x^2-a^2)^2$. 24. $(x+1)(x+3)(x^2-4)$.
25. $(x-1)(x+2)(x+6)(x^2-2x+4)$.
26. $(a+1)(a+2)(a+3)(a+4)(a+5)$.

XXXV. 1. $\frac{a}{b}$. 2. $\frac{x}{y}$. 3. $\frac{c}{2a^2b^2}$. 4. $\frac{3}{2a^2b^2}$.
5. $\frac{y^4}{x^2}$. 6. $\frac{3}{4}x^5z^8$. 7. $\frac{3ac^4}{5b^2x^3}$. 8. $\frac{5cd^3}{6a^3b}$.
9. $\frac{3a^4x^2z^3}{5b^4y^2}$. 10. $\frac{5a^3c^2}{7y}$. 11. $\frac{2c^4z^4}{3a^4x^4}$. 12. $\frac{3ac^3x^2}{4b^2y^2z^2}$.

XXXVI. 1. $\frac{2b}{a+b}$. 2. $\frac{y^2}{1-y^2}$. 3. $\frac{a-b}{a+b}$. 4. $\frac{x}{x-a}$.

5. $\frac{x-1}{x+1}$. 6. $\frac{1-y}{1+y}$. 7. $\frac{x}{x-2}$. 8. $\frac{x^2}{x^2-1}$.

9. $\frac{4}{x+4}$. 10. $\frac{2x}{1+2x}$. 11. $-\frac{1}{x+2}$. 12. $-\frac{1}{a+3}$.

13. $-\frac{x^2}{x^2+a^2}$. 14. $-\frac{x^2+a}{a}$. 15. $\frac{x^3}{x^2+a^2}$.

16. $-\frac{x^2y^2}{x^2y^2+a^2}$. 17. $\frac{5a}{x+3a}$. 18. $-\frac{x}{x+4a}$.

19. $\frac{a-x}{a+x}$. 20. $\frac{a^2+b^2}{a^2-b^2}$. 21. $\frac{x+1}{x^2+x+1}$.

22. $\frac{x^2+1}{x^4+x^2+1}$. 23. $\frac{x-2}{x-4}$. 24. $\frac{1-2a}{1-4a}$.

25. $\frac{x-4}{x+11}$. 26. $\frac{1-4y^2}{1+11y^2}$. 27. $\frac{x-y}{x+4y}$.

28. $\frac{1-a^2b^2}{1+4a^2b^2}$. 29. $\frac{a^2+ax+x^2}{a^2-ax+x^2}$. 30. $\frac{a-b}{a+b}$.

31. $\frac{a^2+b^2}{a^2+ab+b^2}$. 32. $\frac{a^2-ab+b^2}{a^2+b^2}$.

XXXVII. 1. $\frac{x+2}{2x+1}$. 2. $\frac{3x-5}{x^2-3x+2}$. 3. $\frac{x^2+5x+10}{x^3+2x^2+3x+6}$.

4. $\frac{x+1}{x}$. 5. $\frac{2x^2+3ax+7a^2}{x^2-6ax+2a^2}$. 6. $\frac{x^2+2x+3}{3x^2+2x+1}$.

7. $\frac{x-2}{x^2+1}$. 8. $\frac{x^2-2x+1}{x^2-3x+1}$. 9. $\frac{x+2}{2x+1}$.

10. $\frac{x^2-5x+1}{x^2+4x-5}$.

XXXVIII. 1. $\frac{20}{30x}$, $\frac{24}{30x}$, $\frac{7}{30x}$.

2. $\frac{12bc}{24abcx}$, $\frac{4ca}{24abcx}$, $\frac{3ab}{24abcx}$. 3. $\frac{a^2}{abc}$, $\frac{b^2}{abc}$, $\frac{c^2}{abc}$.

4. $\frac{b^2c^2}{abc}$, $\frac{c^2a^2}{abc}$, $\frac{a^2b^2}{abc}$. 5. $\frac{2}{2(x+1)}$, $\frac{5}{2(x+1)}$.

6. $\frac{9}{6(x-1)}$, $\frac{8}{6(x-1)}$. 7. $\frac{20}{24(x+2)}$. $\frac{9}{24(x+2)}$.

8. $\frac{2(x-1)}{2(x^2-1)}$, $\frac{3(x-1)}{2(x^2-1)}$, $\frac{10}{2(x^2-1)}$.

9. $\frac{18(x+1)}{15(x^2-1)}$, $\frac{10(x-1)}{15(x^2-1)}$, $\frac{60}{15(x^2-1)}$.

10. $\frac{a(x+a)}{x^2-a^2}$, $\frac{-x(x+a)}{x^2-a^2}$, $\frac{a^2}{x^2-a^2}$, $\frac{-x^2}{x^2-a^2}$.

11. $\frac{24a(a+b)}{12(a^2-b^2)}$, $\frac{-6b(a+b)}{12(a^2-b^2)}$, $\frac{9a^2}{12(a^2-b^2)}$, $\frac{-10b^2}{12(a^2-b^2)}$.

12. $\frac{(x+1)^2}{(x+1)^3}$, $\frac{2x(x+1)}{(x+1)^3}$, $\frac{3x^2}{(x+1)^3}$.

13. $\frac{x-c}{(x-a)(x-b)(x-c)}$, $\frac{x-a}{(x-a)(x-b)(x-c)}$, $\frac{x-b}{(x-a)(x-b)(x-c)}$.

14. $\frac{b-c}{(a-b)(a-c)(b-c)}$, $\frac{-(a-c)}{(a-b)(a-c)(b-c)}$, $\frac{a-b}{(a-b)(a-c)(b-c)}$.

15. $\frac{a^2+1}{a+1}$. 16. $\frac{a^2}{a-x}$. 17. $\frac{x^2}{x-2y}$.

18. $\frac{a^2-b^2}{a^2b^2}$. 19. $-\frac{2a}{15}$. 20. $\frac{11b}{6}$.

21. $\frac{17a-19b}{20}$. 22. $-\frac{x+17y}{12}$. 23. $\frac{28x-29y}{12}$.

24. $\frac{x+6}{12}$. 25. $\frac{5x-2y}{12}$. 26. 1.

27. 1. 28. $\frac{1}{x+a}$. 29. $-\frac{1}{1+x}$.

30. $-\frac{1}{2+x}$. 31. $\frac{1}{x-3}$. 32. $\frac{1}{(x-3)(x-2)}$.

33. $-\frac{y}{(x-4y)(x-5y)}$. 34. $\frac{2x}{(3x-2y)(5x-2y)}$. 35. $\frac{4x}{1-x^2}$.

36. $\frac{8ab}{a^2-4b^2}$. 37. $\frac{x}{(x-2)^2}$. 38. $\frac{a^2}{(a-2b)^2}$.

39. $\frac{4y}{x-y}$. 40. $\frac{a^2+b^2}{a^2-b^2}$. 41. $\frac{1}{a}$.

42. $\frac{8(x+6)}{x^4-16}$. 43. $\frac{x+3}{x^4-1}$. 44. $\frac{1}{1-9x^2}$.

45. $\frac{4a^4}{a^4-x^4}$. 46. $\frac{8a^8}{a^8-x^8}$.

47. $\frac{2x^2+10x+14}{(x+1)(x+2)(x+3)(x+4)}$. 48. $\frac{2}{x(x^2-1)}$.

49. $\frac{2}{a(a+1)(a+2)}$. 50. $\frac{48}{(x^2-1)(x^2-9)}$.

51. $\frac{6}{a(a+1)(a+2)(a+3)}$. 52. $\frac{24}{x(x^2-1)(x^2-4)}$.

53. $\frac{24}{a(a+1)(a+2)(a+3)(a+4)}$. 54. 0. 55. 0.

56. $\frac{2}{(x-1)(x-2)(x-3)}$. 57. 0. 58. $\frac{x}{(x-2a)^2}$.

59. $\frac{1}{a+2}$. 60. $2\frac{x+ay}{x-ay}$. 61. $\frac{4x^2+3}{x^2-1}$.

62. $\frac{3-x}{1-x}$. 63. $\frac{2}{x-3y}$. 64. 1. 65. 1.

XXXIX. 1. $\frac{1}{2}$. 2. $\frac{ab}{4c^2}$. 3. $\frac{ad}{c^2}$.

4. $\frac{2ac}{3b^2}$. 5. 1. 6. 1. 7. $\frac{b^2}{c^2}$.

8. $\frac{a^4}{c^4}$. 9. $\frac{8}{abc}$. 10. xyz. 11. x^2.

12. 1. 13. $\frac{2xy}{3ab}$. 14. $\frac{xy}{b^2z}$. 15. $\frac{1}{xy}$.

16. $\frac{b}{a^3}$. 17. $\frac{x^2}{(x+3)(x-2)}$. 18. $\frac{x-y}{x+2y}$.

19. x^3. 20. $\frac{a}{b}$. 21. $\frac{x-1}{x-4}$.

22. $\frac{1}{(x+1)(x+2)(x+3)}$. 23. 1. 24. $\frac{(x-1)(x-5)}{(x-2)(x-4)}$.

25. $\frac{x^2+ax+a^2}{x-2a}$. 26. $\frac{a^2+ax+x^2}{a^2-ax+x^2}$. 27. 1.

28. $\frac{1}{(a-b)^2}$. 29. 1. 30. $\frac{(y+z-x)^2}{(z+x-y)^2}$.

31. $\frac{x^2+y^2}{(x+y)^2}$. 32. $\frac{1}{x^2+y^2}$.

XL. 1. $\frac{5(x+a)}{(x-2a)^2}$. 2. $\frac{1}{2x^2-1}$. 3. 1. 4. 0.

5. 0. 6. $\frac{2}{y}$. 7. 1. 8. $x-1$. 9. $\frac{3(4x+5)(3x+4)}{(2x+3)(5x+6)}$.

10. $\frac{1+x^4}{x(1+x^2)}$. 11. $\frac{1}{3x^3}$. 12. $\frac{x^2(x-1)}{x+1}$.

13. $\frac{4}{4x-a+3}$. 14. 1.

XLI. 1. $\frac{2x^2}{3z^2}$. 2. $\frac{3a}{4cx}$. 3. $\frac{a^2b^2c^2}{x^2y^2z^2}$.

4. $\frac{x-5y}{x(x-4y)}$. 5. $\frac{x(x-11y)}{y(x-9y)}$. 6. $\frac{a^2+ab+b^2}{a^2+b^2}$.

7. $a^4+a^2b^2+b^4$. 8. $\frac{x-1}{x^2-3}$. 9. $\frac{3x^2-4x+2}{2x^2-3x+5}$.

10. $\frac{2x^2-xy+3y^2}{3x^2-xy+2y^2}$. 11. $\frac{x-4y}{x^2+4y^2}$. 12. $\frac{x+19y}{15}$.

13. $\frac{57x-66y}{20}$. 14. $\frac{x}{16-x^2}$. 15. $\frac{x}{9-x^2}$.

16. $-\frac{2xy}{(x-2y)^2}$. 17. $\frac{a^2}{(a-4b)^2}$.

18. $\frac{2a^2+22a+62}{(a+4)(a+5)(a+6)(a+7)}$. 19. $\frac{162}{a(a+3)(a+6)(a+9)}$.

20. $\frac{x^2+xy+y^2}{x-3y}$. 21. $\frac{x+y}{x+2y}$. 22. $\frac{(a+b)^2}{(a-b)^2}$.

23. $\frac{(x-5)^2}{(x-8)(3x-8)}$. 24. $\frac{x^2-4xy+y^2}{x^2+y^2}$. 25. $\frac{3}{x(x^3-1)}$.

26. $\frac{3b^2}{(3a+b)(3a-b)(3a+2b)}$. 27. $\frac{1}{4a-b}$. 28. $\frac{1}{a+b}$.

29. m. 30. 1. 31. $\frac{4y}{x+y}$. 32. $\frac{a-b}{a}$.

33. $\frac{1}{x+y}$. 34. $\frac{a^2-9b^2}{a^2-25b^2}$. 35. $\frac{a^2-5b^2}{a^2-9b^2}$.

36. $\frac{1}{(x-1)(x-2)(x-3)}$. 37. $\frac{12-2x}{(x-2)(x-3)(x-4)(x-5)}$.

38. $\frac{14-2x}{(x-3)(x-4)(x-5)(x-6)}$. 39. x^2-xy+y^2-1.

40. x^2+xy+y^2-1. 41. $\frac{1}{1-x}$. 42. 1. 43. 0.

44. $\frac{xy(x+y)}{x^2+y^2}$. 45. $\frac{(a-b)^2}{8a(a+b)^2}$. 46. 4.

47. $\frac{x+3y}{8xy(x+y)}$. 48. 0. 49. $\frac{(a-b)^2}{(a+b)^2}$.

50. $\frac{(a-b)^3}{(a+b)^3}$. 51. $\frac{336\,(11x+5)}{(x-17)\,(x-9)\,(x+4)\,(x+7)}$.

53. $-\frac{1386}{(x-5)\,(x-7)\,(x+2)\,(x+4)}$.

53. $12 \cdot \frac{2x^2+10x+11}{(x+1)\,(x+2)\,(x+3)\,(x+4)}$. 54. $\frac{a+b+c+3d}{a+b+c+d}$.

55. $\frac{1}{xy\,(a-x)}$. 56. $\frac{a}{xyz}$. 57. $-\frac{a}{b}$. 58. 2.

XLII. 1. 7. 2. 4. 3. $\frac{111}{31}$. 4. -4.

5. $-\frac{1}{3}$. 6. $\frac{11}{13}$. 7. $\frac{29}{31}$. 8. $\frac{1}{59}$.

9. $-\frac{3}{7}$. 10. $\frac{13}{19}$. 11. $-\frac{43}{40}$. 12. $-\frac{23}{19}$.

13. 1. 14. $-\frac{60}{13}$. 15. $\frac{35}{6}$. 16. $-\frac{21}{11}$.

17. -2. 18. $-\frac{5}{2}$. 19. -14. 20. 2.

21. $\frac{12}{5}$. 22. 19. 23. $\frac{1}{29}$. 24. $\frac{23}{6}$.

25. $\frac{3}{2}$. 26. -4. 27. -10. 28. -10.

29. 5. 30. $-\frac{9}{2}$. 31. $\frac{1}{16}$. 32. 1.

33. -2. 34. -2. 35. 1. 36. $\frac{8}{3}$.

37. $\frac{31}{5}$. 38. $-\frac{19}{7}$. 39. $\frac{38}{41}$. 40. $\frac{29}{55}$.

41. -2. 42. -4. 43. -6. 44. 5.

45. $\frac{3}{2}$. 46. 4. 47. -3. 48. $\frac{13}{4}$.

49. -6. 50. 1. 51. $\frac{75}{16}$. 52. -1.

53. $-\frac{5}{11}$. 54. $\frac{1}{2}(a+b)$. 55. $\frac{ab}{b-a}$.

56. $-\frac{ab\,(a+b)}{a^2+b^2}$. 57. $-2\,(a+b+c)$. 58. $\frac{a^2+b^2}{a+b}$.

MISCELLANEOUS EXAMPLES. III.

A. 1. $y^3-6xy^2+6x^2y-x^3$.

3. (i) $(x^2+y^2)\,(x+y)\,(x-y)$, (ii) $(a-b)\,(a+b+2)$,
(iii) $(x-7)\,(x+4)$.

4. x^2+x-4, $\frac{x^2-x+5}{x\,(5x^2+4x+16)}$.

5. (i) 5, (ii) $x=1$, $y=-1$, (iii) $x=-\frac{a+b}{2}$.

6. *A* has £11. 15*s*., *B* has £3. 5*s*.

B. 1. 0, 0.

3. (i) $x(x-2)^2$, (ii) $(2x-1)(x-2)$, (iii) $(x^2+1)(x-1)$.

4. $\frac{2}{x+3}$. 5. (i) $\frac{3}{2}$, (ii) $x=a$, $y=0$. 6. $\frac{8}{50}$.

C. 3. (i) $(1-3x)(1+21x)$, (ii) $3y(x-2y)(x^2+2xy+4y^2)$, (iii) $(x+1)^3(x-1)^3$.

4. $\frac{x-3}{x^2+7x+3}$, $\frac{x}{(x-1)(x-2)(x-3)}$.

5. (i) $\frac{17}{7}$, (ii) $x=\frac{c(b-c)}{a(b-a)}$, $y=\frac{c(a-c)}{b(a-b)}$.

6. A, £70; B, £30.

D. 1. 4. 3. (i) $x(x+3y)(x-2y)$, (ii) $(x-a)(x+a)^2$, (iii) $(x-1)(x+1)(y-1)(y+1)$.

4. (i) $\frac{2(a+x)}{a-x}$, (ii) $\frac{2}{(x-4)(x-5)(x-6)}$.

5. (i) $x=5$, (ii) $x=\frac{1}{5}$, $y=-17$. 6. 72.

E. 2. $x^3+x^2y+xy^3+y^3$, $(x+y)^3+2z(x+y)^2+4z^2(x+y)+8z^3$.

3. $(x+5)(5x-1)$, $a(a+b+c)(a-b-c)$.

4. $2a(2a-3b)(2a+3b)(2a-b)$. 6. £240.

F. 1. 59. 2. $x^5-ax^4-a^4x+a^5$.

3. $(x+1)^2(x-1)^2(x^2+x+1)(x^2-x+1)$. 4. $\frac{x}{4x^2-y^2}$.

5. (i) $-\frac{a^2+b^2}{a+b}$, (ii) $x=3$, $y=-1$, $z=0$. 6. 350 acres.

XLIII. 1. 1, -2. 2. 3, 4. 3. -1, -2.

4. $-\frac{1}{2}$, $\frac{1}{2}$. 5. $\frac{1}{3}$, $-\frac{1}{3}$. 6. $\frac{5}{2}$, 4. 7. 0, 2, -3.

8. 0, -2, 4. 9. 0, 3, -4. 10. 0, $\frac{1}{2}$, $-\frac{4}{3}$.

11. 0, $\frac{2}{5}$, $\frac{7}{6}$. 12. 0, 3, $-\frac{7}{3}$. 13. 1, 2, 3, 5.

14. 2, 1, -1, -2. 15. $\frac{1}{3}$, $-\frac{1}{4}$, $\frac{2}{5}$, $-\frac{7}{2}$. 16. $\frac{3}{2}$, $\frac{4}{3}$, $\frac{5}{4}$, $-\frac{6}{5}$.

17. 0, 1. 18. 0, 2. 19. 0, -3. 20. 0, $\frac{3}{2}$.

21. 0, $\frac{5}{2}$. 22. 0, $-\frac{1}{3}$. 23. 0, 5. 24. 0, $\frac{1}{2}$.

25. 0, $\frac{5}{3}$. 26. 0, $-\frac{6}{5}$. 27. 0, b. 28. 0, $\frac{b}{a}$.

29. ± 1. 30. ± 5. 31. ± 4. 32. 0. 33. ± 3.
34. ± 3. 35. ± 1. 36. ± 9. 37. 2, 3.
38. 3, 4, 39. 2, 10. 40. 4, 5. 41. 4, 7.
42. 10, 15. 43. 12, -7. 44. 13, -12. 45. 10, -15.
46. -3, 1. 47. 5, -9. 48. 5, -2.
49. 0, 9, -20. 50. 0, 12, -11.

XLIV. 1. 9, -5, 2. 13, -7. 3. 1, 6.
4. 3, 6. 5. 6, -3. 6. 1, -3. 7. 3, -1.
8. 1, -3. 9. ± 2. 10. ± 3. 11. $-\frac{1}{3}$.
12. $\frac{4}{3}$. 13. 3, 50. 14. 11, -9. 15. 3, $\frac{1}{3}$.
16. $\frac{4}{3}$, -5. 17. $\frac{3}{4}$, -6. 18. $\frac{5}{6}$, -10. 19. $\frac{5}{2}$, $-\frac{5}{4}$.
20. 40, -5. 21. 2, $\frac{1}{19}$. 22. 4, $\frac{2}{17}$. 23. $\frac{5}{17}$, $-\frac{2}{3}$.
24. $\frac{1}{10}$, $\frac{1}{11}$. 25. $\frac{4}{3}$, $-\frac{5}{7}$. 26. $\frac{4}{7}$, $-\frac{5}{3}$. 27. $\frac{2}{3}$, $\frac{3}{2}$.
28. 1, $-\frac{1}{6}$. 29. $\frac{4}{3}$, $\frac{17}{3}$. 30. $\frac{1}{4}$, $\frac{3}{4}$. 31. 3, $-\frac{46}{29}$.
32. 2, $-\frac{69}{29}$. 33. 11, -21. 34. 0, 4. 35. 10, $\frac{18}{5}$.
36. 4, -2. 37. $-\frac{7}{2}$, $-\frac{7}{8}$. 38. a, $2a$.
39. a, $3a$. 40. b, $2a-b$. 41. $-\frac{1}{2}(a-b)$, $-\frac{1}{2}(a+b)$.
42. b, $-2a-b$.

XLV. 1. 2, $\frac{1}{2}$. 2. $\frac{2}{3}$, $\frac{3}{2}$. 3. 6, $\frac{2}{3}$. 4. 3, -7.
5. 3, -2. 6. $-\frac{1}{3}$, $-\frac{8}{3}$. 7. 3, $-\frac{14}{3}$. 8. 2, $\frac{8}{5}$.
9. $2\pm\sqrt{3}$. 10. 5, $\frac{28}{17}$. 11. -1, $-\frac{16}{3}$. 12. -5, -15.
13. -3, $\frac{1}{2}$. 14. 7, $\frac{3}{2}$. 15. 3, $-25{\cdot}7$. 16. 4, $\frac{49}{5}$.
17. 5, $-\frac{1}{6}$. 18. 2. 19. 1, $-\frac{4}{3}$. 20. 4, -1.
21. 2, $\frac{1}{2}$. 22. 4, $-\frac{27}{5}$. 23. 5, $-\frac{33}{13}$. 24. 6, $-\frac{52}{7}$.
25. 3, $-\frac{5}{7}$. 26. 5, $-\frac{7}{3}$. 27. $2\pm 2\sqrt{3}$.
28. $\frac{1}{2}(-1\pm\sqrt{3})$. 29. ± 6. 30. 1. 31. 6.
32. a, $2a$. 33. $\frac{1}{3}\{a\pm\sqrt{(2a^2-b^2)}\}$. 34. $-a$, $\frac{1}{a}$.
35. a, $\frac{1}{a}$. 36. $-b$, $2a-b$. 37. c, $c-2b$.
38. 1, $\frac{a-b}{b-c}$. 39. 1, $-\frac{a+b+c}{a+b}$. 40. $\frac{a}{b}$, $\frac{b}{a}$.
41. $-\frac{b}{c}$, $-\frac{c}{b}$. 42. $\frac{a+b}{a-b}$, $\frac{b-a}{b+a}$. 43. $\frac{b+a}{b-a}$, $\frac{b-a}{b+a}$.

44. $a, -\frac{1}{a}$. 45. $\pm\frac{a}{\sqrt{2}}$. 46. $a+b, \frac{2ab}{a+b}$.

47. $a+b, \frac{ab(a+b)}{a^2+b^2}$. 48. $a+b, \frac{a^2+b^2}{a+b}$. 49. $b, \frac{a^2}{b}$.

50. $-a, -b$. 51. a, b. 52. $c, -\frac{a^2+b^2+ac+bc}{a+b+2c}$.

53. $c, -\frac{ab(2c+a+b)}{ac+bc+2ab}$. 54. $0, \pm\sqrt{ab}$.

55. $0, -\frac{1}{2}(a+b)$.

XLVI. 1. 12. 2. 23. 3. 5. 4. 9. 5. 1.
6. -1. 7. $\frac{113}{66}$. 8. $2, -1$. 9. $5, -4$.
10. 3, 8. 11. 7, 2. 12. 2, 11. 13. 5, 9.
14. 4, 0. 15. $4, \frac{8}{3}$. 16. $9, \frac{1}{4}$. 17. 1, 6.
18. $1, -\frac{1}{2}$. 19. $5, \frac{220}{49}$. 20. 15. 21. $16, \frac{9}{4}$.
22. $\frac{1}{4}, \frac{9}{4}$. 23. $0, -4$. 24. $5, \frac{3}{4}$. 25. 7.
26. $6, -\frac{2}{9}$. 27. $-\frac{1}{2}$. 28. 5. 29. 6.
30. $6, -\frac{18}{25}$. 31. $\frac{1}{3}, \frac{1}{19}$. 32. $3, -\frac{59}{39}$. 33. 2, 3.
34. 0. 35. $\frac{4}{5}$. 36. $\frac{25}{13}$. 37. $2, -\frac{43}{10}$.
38. a, b. 39. $0, 4(a+b)$. 40. $-a, -b$.
41. $-\frac{(a-b)^2}{8(a+b)}$.

XLVII. 1. $x^2-4=0$. 2. $x^2+x-12=0$. 3. $x^2+5x+6=0$.
4. $x^2-\frac{1}{4}=0$. 5. $12x^2-x-1=0$. 6. $6x^2+5x+1=0$.
7. $x^2-3x=0$. 8. $x^2+4x=0$. 9. $x^3-2x^2-15x=0$.
10. $x^2-2=0$. 11. $x^2-5=0$. 12. $x^3-3x=0$.
13. $x^2-4x+1=0$. 14. $x^2-10x+18=0$.
15. $x^2-2ax+a^2-b=0$.

16. (i) 1. (ii) -2. (iii) $-\frac{7}{3}$. (iv) $-\frac{1}{5}$. (v) $\frac{4c}{9a}$.

17. (i) 0. (ii) -3. (iii) 5. (iv) $\frac{1}{2}$. (v) $-\frac{7b}{6a}$.

20. 12. 21. $14p^2$. 23. $a=\frac{4}{3}$. 24. $a=3$ or $a=-5$.
25. $a=8$. 29. $2x^2-23x+11=0$. 30. $9x^2-49x+49=0$.
32. $ac-b^2=0$.

XLVIII. **1.** $\pm 1, \pm 2$. **2.** $\pm 2, \pm\sqrt{-2}$. **3.** $\pm 1, \pm 3$.

4. $\pm 3, \pm\sqrt{-2}$. **5.** $\pm 1, \pm 3$. **6.** $\pm 5, \pm 2$.

7. $\pm a, \pm\frac{1}{a}$. **8.** $\pm a, \pm\frac{\sqrt{-1}}{a}$.

9. $\pm a, \pm a\sqrt{-1}, \pm\frac{1}{a}, \pm\frac{\sqrt{-1}}{a}$. **10.** $2, -1, 3, -2$.

11. $-2, 1, 4, -5$. **12.** $-3, 2, -\frac{1}{2}\pm\frac{1}{2}\sqrt{-27}$.

13. $\frac{1}{2}\pm\frac{\sqrt{5}}{2}$. **14.** $1, \frac{1}{4}\pm\frac{1}{4}\sqrt{-15}$. **15.** $2, \frac{1}{8}\pm\frac{1}{8}\sqrt{-15}$.

16. $-\frac{1}{2}\pm\frac{1}{2}\sqrt{5}, -\frac{1}{2}\pm\frac{1}{2}\sqrt{-1}$. **17.** $1, -2, -\frac{1}{2}\pm\frac{1}{2}\sqrt{-19}$.

18. $1, -3\pm 2\sqrt{2}$. **19.** $0, -8, 1, 3$.

20. $-3, 2, -\frac{1}{2}\pm\frac{1}{2}\sqrt{-27}$. **21.** $\frac{7}{2}, -\frac{3}{2}, 1\pm\sqrt{5}$.

22. $3, -6, -\frac{3}{2}\pm\frac{1}{2}\sqrt{125}$. **23.** $0, -3, -\frac{3}{2}\pm\frac{1}{2}\sqrt{14}$.

24. $2, -8, -3\pm\sqrt{45}$. **25.** $7, -\frac{3}{2}, \frac{11}{4}\pm\frac{1}{4}\sqrt{243}$.

26. $\frac{1}{2}, -\frac{1}{2}, 2, 3$. **27.** $-8, 5, -\frac{3}{2}\pm\frac{1}{2}\sqrt{221}$.

28. $1, -\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$. **29.** $1, -\frac{1}{2}\pm\frac{1}{2}\sqrt{17}$. **30.** $3, 5, -8$.

31. $3, \pm\sqrt{7}$. **32.** $-1, 4, -1$. **33.** $2, \frac{1}{2}\pm\frac{1}{10}\sqrt{165}$.

34. $x=-3$. **35.** $x=-5$. **36.** (i) $1, 4, -5$.

(ii) $3, -3, -2$. (iii) $2, 4, -5$. (iv) $10, -2, -5$.

(v) $2, 3-5$. (vi) $1, -4, -\frac{1}{2}$. (vii) $-1, \frac{3}{2}, -\frac{3}{2}$.

XLIX. **1.** $5, 1$. **2.** $3, -7;\ 7, -3$. **3.** $3, \frac{1}{3};\ \frac{1}{3}, 3$.

4. $6, -4;\ -6, 4$. **5.** $5, 2;\ -5, -2$. **6.** $12, 3;\ -5, 20$.

7. $3, 1;\ \frac{5}{4}, -\frac{5}{2}$. **8.** $7, -2;\ -\frac{48}{91}, -\frac{593}{91}$.

9. $1, 1;\ \frac{117}{73}, \frac{7}{73}$. **10.** $2, -3;\ \frac{374}{313}, -\frac{1001}{313}$.

11. $\frac{1}{2}, \frac{3}{2};\ \frac{4}{3}, \frac{2}{3}$. **12.** $1, 3;\ \frac{21}{5}, \frac{7}{5}$. **13.** $2, 3;\ 3, 2$.

14. $3, 2;\ \frac{2}{5}, -\frac{3}{5}$. **15.** $1, -2;\ 6, 8$. **16.** $\frac{1}{2}, \frac{7}{2};\ -\frac{5}{3}, -\frac{14}{9}$.

17. $\frac{7}{2}, \frac{23}{7}$. **18.** $1, 3;\ \frac{5}{4}, 2$. **19.** $6, 7;\ -\frac{13}{2}, \frac{53}{4}$.

20. $-1, -3;\ \frac{8}{7}, \frac{18}{49}$.

L. **1.** $0, \pm 5;\ \pm 6, \pm 3$*. **2.** $\pm 3, \pm 2;\ 0, \pm\sqrt{34}$.

3. $\pm 3, \pm 4;\ \pm 7\frac{1}{2}, \mp\frac{1}{2}$. **4.** $\pm 9, \pm 2;\ \pm\frac{7}{\sqrt{2}}, \mp\frac{11}{\sqrt{2}}$.

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* The upper signs to be taken together and the lower signs to be taken together.

5. $\pm 4, \pm 2;\ \pm 6\sqrt{-2}, \mp 8\sqrt{-2}$. **6.** $\pm 5, \pm 1;\ \pm 4, \pm \frac{1}{2}$.
7. $\pm 14, \mp 8;\ \pm 1, \pm 5$. **8.** $\pm 1, \pm 2;\ \pm 7\sqrt{-\frac{1}{2}}, \mp 3\sqrt{-\frac{1}{2}}$.
9. $\pm 5, \pm 2$. **10.** $\pm 2, \pm 1$. **11.** $\pm 3, \pm 5;\ \pm 36, \mp \frac{23}{2}$.
12. $\pm 5, \pm 3;\ \pm 4\sqrt{2}, \pm\sqrt{2}$. **13.** $\pm 1, \pm 2;\ \pm\sqrt{3}, 0$.
14. $\pm 1, \pm 3;\ \pm 5, \pm 3$. **15.** $\pm 8, \mp 2;\ \pm \frac{2}{9}\sqrt{21}, \mp \frac{8}{9}\sqrt{21}$.
16. $\pm 2, \mp 1;\ \pm \frac{18}{7}, \mp \frac{1}{7}$. **17.** $\pm 1, \mp 2;\ \pm 5, \pm 6$.
18. $\pm 5, \pm 1;\ \pm 16, \mp \frac{9}{2}$. **19.** $\pm 1, \pm 2;\ \pm \frac{1}{7}, \pm \frac{18}{7}$.
20. $\pm 3, \pm 2;\ \pm \dfrac{31}{\sqrt{145}}, \mp \dfrac{8}{\sqrt{145}}$. **21.** $3, -4;\ -4, 3$.
22. $3, -6;\ -6, 3$. **23.** $-4, 6$. **24.** $4, 15;\ 6, 10$.
25. $3 \pm \sqrt{6}, 3 \mp \sqrt{6}$. **26.** $3, -\frac{1}{3};\ -1, 1$.

LI. **1.** $7, 4;\ -4, -7$. **2.** $5, 3;\ -3, -5$. **3.** $4, 3;\ 3, 4$.
4. $5, -4;\ -4, 5$. **5.** $\pm 3, \pm 1;\ \pm 1, \pm 3$. **6.** $\pm 3, \pm 3$.
7. $6, 4;\ 4, 6$. **8.** $4, 1;\ -1, -4$. **9.** $\frac{1}{4}, \frac{1}{2};\ -\frac{1}{2}, -\frac{1}{4}$.
10. $-\frac{1}{4}, \frac{1}{9}$. **11.** $\pm\sqrt{2}, \pm\sqrt{2};\ \pm \frac{1}{6}\sqrt{59}, \mp \frac{1}{4}\sqrt{59}$.
12. $\pm \frac{1}{2}, \mp \frac{1}{2}$. **13.** $-3, 4;\ 4, -3;\ \frac{1}{2} \pm \frac{1}{2}\sqrt{5}, \frac{1}{2} \mp \frac{1}{2}\sqrt{5}$.
14. $-1, 6;\ 6, -1;\ \frac{5}{2} \pm \frac{1}{2}\sqrt{17}, \frac{5}{2} \mp \frac{1}{2}\sqrt{17}$. **15.** $-\frac{3}{4}, -\frac{15}{4};\ 8, 10$.
16. $3, 0;\ -7, 5$. **17.** $\frac{3}{2}, \frac{2}{3}$. **18.** $\frac{6}{7}, \frac{7}{12};\ \frac{12}{7}, \frac{7}{6}$.
19. $0, 0;\ 1, 2;\ \frac{15}{22}, \frac{9}{22}$. **20.** $-1, 0;\ 2, 1;\ \frac{1}{2}, \frac{1}{8}$.
21. $5, -3$. **22.** $5, 4;\ 4, 5;\ \frac{1}{2}(-9 \pm \sqrt{161}), \frac{1}{2}(-9 \mp \sqrt{161})$.
23. $\pm 3, \pm 1;\ \pm 1, \pm 3;\ \pm 3\sqrt{-1}, \pm\sqrt{-1};\ \pm\sqrt{-1}, \pm 3\sqrt{-1}$.
24. $1, 1;\ -\frac{5}{2} \pm \frac{1}{2}\sqrt{-7}, -\frac{5}{2} \mp \frac{1}{2}\sqrt{-7}$.
25. $\pm \frac{1}{2}, \pm \frac{1}{3};\ \pm \frac{1}{3}\sqrt{-14}, \mp \frac{1}{2}\sqrt{-14}$.
26. $\pm \frac{1}{2}, \pm 3;\ \pm \frac{1}{2}, \pm \frac{1}{3}$. **27.** $0, 0;\ \dfrac{a^2+b^2}{a}, \dfrac{a^2+b^2}{b}$.
28. $a+b, a+b;\ \dfrac{b}{a}(a-b), \dfrac{a}{b}(b-a)$.
29. a, b, **30.** $\pm 6, \pm \frac{8}{3}, \pm \frac{3}{2}$.
31. $\pm 1, \pm 2, \pm 3;\ \pm\sqrt{-1}, \pm 2\sqrt{-1}, \pm 3\sqrt{-1}$.
32. $\pm \frac{8}{5}, \pm \frac{12}{5}, \pm 1$. **33.** $\pm 1, \pm 4, \pm 2$. **34.** $0, \pm 1, \pm 2$.

LII. **1.** ±27, ±9. **2.** 7, 11. **3.** ±25 and ±5.
4. 12 and 13. **5.** 38, 42. **6.** 3 or 63.
7. 39 yds., 29 yds. **8.** ±2. **9.** 50. **10.** 18.

11. 18 gals. of water, 9 of wine. **12.** 20. **13.** 20.
14. 75. **15.** $3\frac{9}{17}$. **16.** 4 miles per hour. **17.** 45 and 60.
18. $1\frac{5}{6}$ hrs. **19.** 4 days. **20.** 9*s*. per doz.
21. 9*d*. per doz. **22.** 48. **23.** 7*s*. and 8*s*. 9*d*.
24. 72. **25.** £10, £5, £1000. **26.** $\frac{1}{2}$ and $\frac{1}{3}$, or $\frac{5}{2}$ and $-\frac{5}{3}$.
27. $3\frac{1}{3}$ and $1\frac{2}{3}$ miles an hr. **28.** 578. **29.** 18 in., 8 in.
30. 1500 sq. yds. **31.** 9 and 4. **32.** 210 miles or 144 miles.

MISCELLANEOUS EXAMPLES. IV.

A. **1.** y. **2.** $a^3-125b^3+8c^3+30abc$.
3. $x^2+x(a-2b)+(a^2+3b^2)$.
4. (i) $3(x+y+z)(x-y-5z)$, (ii) $(x-1)(x+1)(y-1)(y+1)$,
(iii) $(x^2z-1)(y^2z-1)$.
5. (i) $\frac{x+6}{2-x-x^2}$, (ii) $\frac{2}{(x-1)(x-2)(x-3)}$.
6. (i) $x=\frac{3ab-2ac-bc}{a+2b-3c}$, (ii) $x=7$ or $-\frac{3}{4}$,
(iii) $x=2$ or $\frac{1}{2}$, $y=\frac{1}{2}$ or 2. **8.** 17 and 18.

B. **1.** (i) 6. **3.** $x^4+2ax^3+3a^2x^2+2a^3x+a^4$.
4. $(64x^6-729)(3x+2)$. **5.** (i) $\frac{2(x+3)}{x^4-1}$, (ii) 1.
6. (i) $x=-\frac{1}{5}$, (ii) $x=5$ or $-6\frac{2}{5}$,
(iii) $x=\pm 2$, $y=\pm 3$, or $x=\pm\frac{5}{\sqrt{2}}$, $y=\mp\frac{1}{\sqrt{2}}$.
8. 840 tickets.

C. **1.** (i) $24a-21b$, (ii) $a^6-6a^4b^2-7a^2b^4-4b^6$.
3. $4xy-6x^2-2y^2$.
4. (i) $(3x+2)(3x+1)$, (ii) $(a+b+c+d)(a-b+c-d)$
$(a+b-c-d)(-a+b+c-d)$.
5. y. **6.** (i) $x=-212$, (ii) $x=y=a$. **7.** 3 and 5.
8. 6*s*. 2*d*., 5*s*. 5*d*.

D. **2.** $3+2x+8x^2+4x^3+x^4$.
5. $(x+1)(x+2)(x+3)(x-7)$, $x=7$.
6. (i) $x=\frac{2}{3}$, $y=3$, (ii) $x=a$ or b,
(iii) $x=2$, $y=1$; $x=6\frac{1}{4}$, $y=-3\frac{1}{4}$. **8.** 10 gallons.

E. 1. $2\frac{5}{6}$. 3. $-3a+3b-18c+6d$.
4. $(6x^2-5ax-6a^2)(2x^2+2ax+3a^2)$. 5. $\dfrac{1}{x+y}$.
6. (i) $x=\dfrac{a+b}{2}$, (ii) $x=0$ or 4,
(iii) $x=\frac{3}{5}$, $y=\frac{2}{3}$; $x=1\frac{1}{2}$, $y=1\frac{2}{3}$. 7. $x=c$ or $\dfrac{a^2}{c}$.
8. 15 miles an hr.

F. 1. $a^3+a^2b+ab^2+b^3$. 3. $1+4x-x^2$. 4. $x-1$, 1.
5. (i) 0, (ii) x. 6. (i) $x=1$, (ii) $x=a$ and $y=b$, (iii) $x=0$ or $\frac{5}{3}$.
7. $7\sqrt{13}$. 8. 169 or 144.

MISCELLANEOUS EQUATIONS.

1. 1. 2. $\frac{11}{5}$, 3. $\frac{7}{2}$. 4. 3, 2. 5. 15 miles, 18 miles.
6. $\frac{23}{22}$. 7. 5. 8. 8. 9. 280, 336. 10. 2*s*. 1*d*.
11. 7. 12. $a-b$. 13. 1. 14. $x=y=1$. 15. 310.
16. 11. 17. 2, 12. 18. $-\frac{3}{5}$, $-\frac{5}{3}$. 19. 6, 7.
20. 51, 22, 17. 21. 5. 22. $\dfrac{bc}{a}$. 23. 17, 15.
24. $\frac{2}{3}$ or 6. 25. 100 lbs. 26. 7. 27. 10, 11.
28. $\frac{1}{2}(a-b)$, $\frac{1}{2}(a+b)$. 29. 6, $\frac{4}{3}$. 30. 45 years.
31. $-\frac{1}{5}$. 32. $-\frac{1}{2}$, 3. 33. 3, $-\frac{5}{7}$. 34. $\frac{1}{4}$, $\frac{9}{4}$.
35. 2 hours after starting. 36. 0, or 2 37. -1, -5.
38. 8, $\frac{233}{179}$. 39. 8. 40. 2*s*., 5*s*., 13*s*. 41. $\frac{896}{111}$.
42. 5, 6. 43. 0, -1, -8. 44. 10, 9. 45. 99.
46. 2. 47. $\frac{44}{5}$, -11. 48. 4, $\frac{9}{19}$. 49. $\frac{1}{2}$.
50. 1 hour, 45 minutes, 2 hours, 20 minutes. 51. 3.
52. 6, 1. 53. $\pm\sqrt{a^2+1}$.
54. ± 2, ± 3; $\mp\dfrac{32}{\sqrt{345}}$, $\pm\dfrac{41}{\sqrt{345}}$. 55. 6, 7 and 8. 56. 6.
57. 0, 5, $\frac{13}{5}$. 58. $a-b$, $a+b$. 59. ± 3, ∓ 3.
60. 3*s*. 61. c. 62. 1, $-\frac{53}{3}$.
63. 5, -7, $-1\pm\frac{1}{2}\sqrt{69}$. 64. ± 4, ∓ 2; $\pm\frac{1}{2}$, ± 5.
65. 29. 66. 1. 67. $a\dfrac{a^3+b^3}{a^2+ab+b^2}$, $b\dfrac{a^3+b^3}{a^2+ab+b^2}$.

68. $a+b$. 69. $\pm 1, \mp 2; \pm 3, \pm 2$. 70. 50. 71. $-\frac{8}{5}$.

72. $-a, \frac{c^2}{a}$. 73. 0, 8. 74. 5, 7; $\frac{13}{4}, -\frac{7}{4}$.

75. 52 half crowns, 74 shillings, 24 sixpences. 76. $\frac{19}{68}$.

77. $3, -\frac{1}{2}, \frac{1}{5} \pm \frac{1}{5}\sqrt{-14}$. 78. $0, 0; 2a, 2b$.

79. $6, 6; -4, -4; -1 \pm \sqrt{21}, -1 \mp \sqrt{21}$.

80. 32 ft. long, 24 ft. wide, 10 ft. high. 81. $\frac{23}{22}$.

82. $a, b, \frac{1}{2}(a+b)$. 83. 7, 6.

84. $4, 2; 2, 4; 3 \pm \sqrt{21}, 3 \mp \sqrt{21}$. 85. 30. 86. 3.

87. $\frac{1}{2}(a-1), \frac{1}{2}(1-b)$. 88. $\frac{1}{3}, \frac{1}{19}$. 89. $5, -6; -6, 5$.

90. 44 yards, 55 yards. 91. 0, 7. 92. $-1, 1, 1$.

93. a, b. 94. $8, 6; 6, 8; 0, -2; -2, 0$.

95. 50 miles an hour, and 30 miles an hour.

96. $3a-b, 3b-a$. 97. $6, \frac{2}{13}$. 98. $-3, -3 \pm \sqrt{5}$.

99. $4, 3; 3, 4; -\frac{19}{2} \pm \frac{1}{2}\sqrt{313}, -\frac{19}{2} \mp \frac{1}{2}\sqrt{313}$. 100. 15 miles.

LIII. 1. a^{15}. 2. a^{15}. 3. $-a^6$. 4. a^6.

5. $16a^{20}$. 6. $-243a^{20}$. 7. a^4b^8. 8. $a^{15}b^{20}$.

9. $-a^7b^{28}$. 10. $-27a^{21}b^{15}c^3$. 11. $a^4b^8c^{20}$.

12. $-125a^6b^9c^{12}$. 13. $\frac{a^6}{b^6}$. 14. $-\frac{a^{15}}{b^5c^{10}}$.

15. $\frac{a^6}{b^{12}c^{18}}$. 16. $4a^8+12a^4b^3+9b^6$. 17. $a^{10}-4a^5b^4+4b^8$.

18. $a^2x^8-2abx^4y^3+b^2y^6$. 19. $a^4-4a^3b+4a^2b^2$.

20. $a^6+3a^4b^2+3a^2b^4+b^6$. 21. $a^9+3a^6b^3+3a^3b^6+b^9$.

22. $8a^6-36a^4b^2+54a^2b^4-27b^6$.

23. $27a^6-54a^4b^2+36a^2b^4-8b^6$.

24. $a^4+b^4+c^4+2a^2b^2+2a^2c^2+2b^2c^2$.

25. $a^6+4b^6+9c^6-4a^3b^3+6a^3c^3-12b^3c^3$.

26. $a^4+16b^4+9c^4-8a^2b^2-6a^2c^2+24b^2c^2$.

27. $x^4-6x^3-3x^2+36x+36$. 28. $9x^4-6x^3-29x^2+10x+25$.

29. $4x^4+20x^3+21x^2-10x+1$. 30. $9x^4-36x^3+72x+36$.

31. $1+2x+3x^2+4x^3+3x^4+2x^5+x^6$.

32. $x^6-2x^5+3x^4-4x^3+3x^2-2x+1$.

33. $x^6+2x^5-3x^4-8x^3+8x+4$.

34. $a^2+4b^2+9c^2+16d^2+4ab+6ac+8ad+12bc+16bd+24cd$.

35. $4a^2+b^2+c^2+4d^2-4ab+4ac-8ad-2bc+4bd-4cd$.

36. $x^6+3x^5+6x^4+7x^3+6x^2+3x+1$.
37. $x^6-3x^5+9x^4-13x^3+18x^2-12x+8$.
38. $27x^6-135x^5+252x^4-215x^3+84x^2-15x+1$.

LIV. 1. $3x-5y$. 2. $5x^2-3y^2$. 3. $2x^2-3y^2$.
4. $2x^5-3y^3$. 5. x^4-3y^4. 6. $3x^6-y^3$.
7. $\frac{1}{2}x^3-\frac{1}{3}y^3$. 8. $\frac{1}{3}x^4y^3-\frac{1}{2}$. 9. $5x^4y^3-4a^2b^3$.
10. $\frac{x^2}{a}+4ay^3$. 11. $3\frac{bx^2}{a}-4\frac{ay^2}{b}$. 12. $\frac{3}{x^3}-7a^5$.
13. $a+2b+3c$. 14. $-2a+b+3c$. 15. $2a^2+b^2-c^2$.
16. $5a^2-3b^2+2c^2$.

LV. 1. x^2+x+1. 2. $2x^2-2x-1$. 3. $3x^2-6x-6$.
4. $1-\frac{1}{2}xy-2x^2y^2$. 5. $2x^2+x-\frac{1}{4}$. 6. $x^2-x+\frac{1}{4}$.
7. x^2+xy+y^2. 8. $4-12x+9x^2$. 9. $1+2x+3x^2$.
10. $2x^2-x+\frac{1}{2}$. 11. $1-2x-2x^2$. 12. x^3-2x^2+x-2.
13. $3x^3-2x^2+3x+2$. 14. $x^3-11x+17$.
15. $a-\frac{1}{2}x+4$. 16. x^4+x^3-2x-4. 17. $4x-12+\frac{9}{x}$.
18. $x^3-3x+\frac{3}{x}-\frac{1}{x^3}$. 19. $x^2+x(y+z)+yz$.
20. $2(yz+zx+xy)$.

LVI. 1. 4. 2. $\frac{1}{8}$. 3. $\frac{1}{4}$. 4. 5. 5. $\frac{125}{8}$.
6. $\frac{16}{9}$. 7. $\frac{1}{81}$. 8. $\frac{1}{1000000}$. 9. 1000.
10. $\frac{625}{81}$. 11. $a^{\frac{1}{3}}$. 12. a^2. 13. $a^{\frac{1}{3}}$. 14. $a^{\frac{3}{5}}$.
15. $a^{\frac{1}{3}}b^{\frac{1}{3}}$. 16. ab^2. 17. a^3b^{-1}. 18. ab^{-2}.
19. $a^{-1}b^{\frac{1}{4}}$. 20. a^6b^{-1}. 21. a^{-3}. 22. a.
23. $a^{\frac{1}{2}}$. 24. 1. 25. 1. 26. 1. 27. 1.
28. $y^{\frac{1}{3}}$. 29. x^xy^y. 30. $x^xy^yz^z$. 31. $a^{-\frac{1}{3}}b^{\frac{127}{42}}$.
32. $a^{\frac{1}{2}}x^{-\frac{1}{12}}y^{\frac{13}{12}}$. 33. x^{3q}. 34. 1. 35. 1.
36. a. 37. ay. 38. $a^{\frac{13}{6}}x^{\frac{13}{6}}$. 39. a.
40. $a^{\frac{13}{5}}$. 41. $a^{\frac{1}{24}}$. 42. $a^{\frac{1}{15}}$. 43. $a^{\frac{17}{6}}b^{-\frac{8}{3}}$.
44. b^{-4}. 45. $\sqrt{a}-\frac{1}{a^2}$. 46. $\frac{1}{a^4\cdot\sqrt[4]{b}}$.
47. $\frac{\sqrt[3]{a^2}}{b}-\frac{1}{\sqrt[3]{a^2}\cdot\sqrt{b}}$. 48. $\frac{1}{a^3\cdot\sqrt{b}}+\frac{\sqrt[3]{a^2}}{9\sqrt[3]{b^2}}$.

LVII. 1. $x-y$. 2. x^3-y^3. 3. $x^{\frac{1}{2}}-1$.
4. $x^{\frac{2}{3}}-y^{\frac{2}{3}}$. 5. $x-1$. 6. $x+y$. 7. a^2-b^2.
8. $a^{\frac{8}{3}}+a^{\frac{4}{3}}b^{\frac{4}{3}}+b^{\frac{8}{3}}$. 9. x^2-1. 10. $x^{\frac{5}{2}}-x^{\frac{3}{2}}$.
11. $x^n+2x^{\frac{n}{2}}+3+2x^{-\frac{n}{2}}+x^{-n}$.
12. $x^{2n}y^{-2n}+2x^ny^{-n}+3+2x^{-n}y^n+x^{-2n}y^{2n}$.
13. $a+b+c-3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$. 14. $\frac{1}{16}a^2-\frac{1}{81}b^2$. 15. $243x+y^{\frac{5}{3}}$.
16. $a^{\frac{1}{2}}+b^{\frac{1}{2}}$. 17. $x^{\frac{3}{2}}+y^{\frac{3}{2}}$. 18. $x^n+x^{\frac{n}{2}}y^{\frac{n}{2}}+y^n$.
19. $x^{\frac{4}{3}}-x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}}$. 20. $x^{\frac{4}{5}}+3x^{\frac{3}{5}}y^{\frac{1}{3}}+9x^{\frac{2}{5}}y^{\frac{2}{3}}+27x^{\frac{1}{5}}y+81y^{\frac{4}{3}}$.
21. $y^3+b^{\frac{1}{2}}y^{\frac{5}{2}}-b^{\frac{3}{2}}y^{\frac{3}{2}}+b^{\frac{5}{2}}y^{\frac{1}{2}}+b^3$.
22. $x^{\frac{1}{2}}y^{-\frac{1}{2}}+x^{\frac{1}{4}}y^{-\frac{1}{4}}+x^{-\frac{1}{4}}y^{\frac{1}{4}}+x^{-\frac{1}{2}}y^{\frac{1}{2}}$.
23. $a^2-ab^{\frac{2}{3}}+b^{\frac{4}{3}}$. 24. $a^{\frac{2}{3}}-a^{-\frac{2}{3}}$.
25. $x^{\frac{2}{3}}+2x^{\frac{1}{2}}+1+2x^{-\frac{1}{2}}+x^{-\frac{2}{3}}$.
26. $x^2y^{-\frac{12}{5}}-x^{\frac{4}{3}}y^{-\frac{8}{5}}+x^{\frac{2}{3}}y^{-\frac{4}{5}}-1+x^{-\frac{2}{3}}y^{\frac{4}{5}}-x^{-\frac{4}{3}}y^{\frac{8}{5}}+x^{-2}y^{\frac{12}{5}}$.
27. $a^2c^{\frac{5}{2}}$. 28. a^3-64b^2. 31. $4x^2+3x+2-3x^{-1}$.
32. $\frac{8}{1-x^2}$. 33. $2b$.
34. (i) $x^{\frac{1}{3}}+1$. (ii) $2x^{\frac{1}{3}}-y^{\frac{1}{3}}$. (iii) $a^{\frac{1}{2}}x^{\frac{1}{3}}-a^{\frac{1}{3}}x^{\frac{1}{2}}$.
35. $2xa^{-1}-3+4x^{-1}a$. 36. $5xy^{-1}-2+\frac{1}{2}x^{-1}y$.
37. $x^{\frac{4}{5}}-a^{-\frac{3}{5}}x^{\frac{7}{5}}+a^{\frac{4}{5}}$. 38. $x^{\frac{1}{3}}-2x^{\frac{1}{2}}+x^{\frac{5}{6}}$.
39. (i) $1,\ \frac{1}{2}\{-1\pm\sqrt{-3}\}$. (ii) 36, 4. (iii) 81, 1. (iv) 1, 27.
(v) $\frac{243}{32},\ -\frac{1}{32}$.

LVIII. 1. $7\sqrt{3}$. 2. $12\sqrt{2}$. 3. $13\sqrt{5}$. 4. $3\sqrt{5}$.
5. $\sqrt{7}$. 6. $5\sqrt{13}$. 7. $8\sqrt[3]{5}$. 8. 0. 9. $\sqrt[3]{7}$.
10. $4\sqrt{2}$. 11. $13\sqrt{3}$. 12. 0. 13. $\frac{2}{3}\sqrt{2}$.
14. 30. 15. $12\sqrt{2}$, 16. 6. 17. 6. 18. $90\sqrt{3}$.
19. $6\sqrt[3]{4}$. 20. 30. 21. 6. 22. $10\sqrt[6]{40}$.
23. $4\sqrt[6]{2}$. 24. $\frac{3}{5}$. 25. $\frac{3}{4}$. 26. 8. 27. $7\sqrt[3]{9}$.
28. $\sqrt[3]{5\frac{1}{5}}$. 29. $\sqrt{50},\ \sqrt[3]{344},\ \sqrt[4]{2402}$. 30. 3.
31. $\sqrt{15}-6$. 32. $19+5\sqrt{6}+6\sqrt{3}+12\sqrt{2}$.
33. $1+2\sqrt[3]{3}+\sqrt[3]{9}-\sqrt[3]{25}$. 34. $10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15}$.
35. 24.

LIX. 1. $\frac{3}{7}\sqrt{7}$. 2. $\frac{2}{5}\sqrt{5}$. 3. $\frac{1}{2}\sqrt{6}$. 4. $\frac{1}{5}\sqrt{30}$.
5. $2\sqrt{2}-2$. 6. $3-2\sqrt{2}$. 7. $5-\sqrt{15}$.
8. $\frac{103+80\sqrt{3}}{71}$. 9. $\frac{19+7\sqrt{15}}{17}$. 10. $\frac{8-7\sqrt{2}}{2}$.
11. $\frac{2+\sqrt{2}-\sqrt{6}}{4}$. 12. $\frac{12+9\sqrt{3}+3\sqrt{5}-6\sqrt{15}}{22}$.
13. $2\sqrt[3]{4}+2$. 14. $\sqrt[3]{81}+1$. 15. 14. 16. 52.
17. $\frac{177+52\sqrt{2}}{161}$. 18. $\frac{26-15\sqrt{3}+3\sqrt{10}}{13}$. 19. $\frac{29}{2}$.
20. 98. 22. $\frac{2+\sqrt{6}+\sqrt{2}}{2}$. 23. 0.
25. $\frac{7a+b+8\sqrt{(a^2-b^2)}}{3a+5b}$. 26. 5·828. 27. 6·636.
28. $\sqrt{5}+1$. 29. $3+\sqrt{7}$. 30. $3-\sqrt{3}$. 31. $5-\sqrt{3}$.
32. $2\sqrt{13}-7$. 33. $6\sqrt{2}+3\sqrt{5}$. 34. $14+2\sqrt{21}$.
35. $2+\frac{1}{\sqrt{2}}$. 36. $4\sqrt{5}$. 37. $4-2\sqrt{3}$.
38. $1+\sqrt{5}+\sqrt{7}$. 39. $\sqrt{(x+1)}+\sqrt{(x-1)}$.
40. $\sqrt{(a+x)}+\sqrt{(a-x)}$. 41. $\sqrt{(2x-3)}+\sqrt{(x+2)}$.
42. $\frac{3-\sqrt{7}}{2}$. 43. $\sqrt{8}-\sqrt{7}$. 44. $2\sqrt{5}$. 45. $\frac{2\sqrt{2}+1}{7}$.
46. $\frac{\sqrt{6}}{3}$. 48. 6.

LX. 1. 7 : 8, 31 : 36, 41 : 48, 5 : 6. 2. 2. 3. -13.
4. $\frac{4}{7}$. 5. 55, 66. 6. 21, 56. 7. $x:y=7:2$.
8. $x:y=1:2$, 9. $\frac{x}{y}=\frac{2}{1}$ or $\frac{3}{1}$. 10. 4 : 7.
11. 12 and 9. 12. 16. 13. 5 : 8, 3 : 5.
14. $4x^2$, $9y^2$. 15. $\frac{11}{16}$. 16. 36, 48.

LXI. 6. a^2b^2. 7. a^2-b^2. 8. a^3, $(a+b)^2$. 20. $\frac{11}{3}$.

LXII. 1. $8\frac{1}{3}$. 2. 5. 6. 6. 7. $6\frac{3}{4}$.
8. 452·38896 sq. feet. 9. 113·076. 10. 160.
11. 64 feet. 13. $\frac{45}{56}$ cubic feet. 14. $a^2-b^2=4c^2$.
15. 1848 cubic feet. 16. 12 inches. 17. 15 miles.

MISCELLANEOUS EXAMPLES. V.

A. 1. $\frac{13}{18}$. $a^2-3ab+b^2$. 3. $1-x^3-x^4$.
4. $\frac{(a-1)(b-1)}{b}$. 5. 3 shillings. 7. 7.

B. 1. $3\frac{3}{7}$. 2. $(a-1)x^3-3ax^2+(a^2-a^3)x-(a^3+2a^2+2a+1)$.
3. (i) $x(x-7y)(x-6y)$, (ii) $(5c-a)(3a+4b+c)$,
(iii) $(x-2y+3)(x-2y-3)$.
4. $\frac{4x}{x+2y}$. 5. (i) 4 or $-\frac{5}{3}$;
(ii) $x=\frac{17}{3}$, $y=\frac{10}{3}$; $x=\frac{277}{32}$, $y=\frac{235}{32}$.
7. $3x^3-2xy^2+5y^3$. 8. 6 and 3.

C. 1. $20b-10a-8c$. 2. $b^2-a^2+\frac{b^4}{a^2}-\frac{a^4}{b^2}$.
3. $3\frac{a^2+b^2}{a+b}$, $3a^2+ab-4b^2$. 5. $a^{\frac{2}{3}}+2a^{\frac{1}{3}}y^{\frac{1}{5}}+3y^{\frac{2}{5}}$.
7. 378 miles.

D. 2. $x^2+(a-b)x-ab$. 3. $6x(x+1)(x-3)(x-4)$.
4. $\frac{1}{2}(y+z-x)^2$.
5. (i) $x=-7$; (ii) ± 2, ± 5; $\pm\frac{7}{2}\sqrt{2}$, $\mp\frac{3}{2}\sqrt{2}$.
6. $16\frac{1}{2}$ feet and $13\frac{3}{4}$ feet. 7. $x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y$.

E. 1. $\frac{1}{3}(x-y)$. 2. $2ax-(3b-4c)y$.
3. (i) $4a^2b^2(b+2a)(b-2a)$, (ii) $(x+3)(x-29)$,
(iii) $(3x+5y)(x-2y)$. 5. 42. 6. $2\sqrt{ax}-2a$.

F. 1. $a^2-b^2-c^2-d^2+2bc+2cd-2bd$. 2. $(a+b+c)^2$.
3. x^2-5x+6; 3 or 2.
5. (i) $x=-\frac{25}{9}$, (ii) $x=2\frac{1}{2}$, $y=-1\frac{1}{6}$; $x=-1\frac{3}{4}$, $y=1\frac{2}{3}$.
6. $30+13\sqrt{6}$, $y^{-\frac{5}{12}}$. 7. $x^6-3x^4+2x^2-1$.

LXIII. 1. (i) 61, (ii) 117, (iii) -75, (iv) $6\frac{5}{9}$, (v) $a-28b$.
2. (i) 72, (ii) 121, (iii) -3, (iv) 366, (v) $40\frac{2}{3}$, (vi) $-19\frac{2}{3}$.
3. 15. 4. $\frac{5}{4}$. 5. 5. 6. 43. 7. 34.
8. -2. 9. 78. 10. 0. 11. 21st.
12. 102nd. 13. 56th. 14. 9th.

15. (i) 10, (ii) 0, (iii) a. **16.** 11, 14, &c.
17. $53\frac{1}{3}$, $56\frac{2}{3}$, &c. **18.** $272\frac{1}{4}$, $275\frac{1}{2}$, &c.
19. $65\frac{1}{2}$, 64, $62\frac{1}{2}$, &c. **20.** $82\frac{1}{3}$, $80\frac{2}{3}$, 79, &c.
21. $4a-5b$, $3a-4b$, $2a-3b$, &c.
22. $\frac{1}{3}a-4\frac{1}{3}b$, $-\frac{1}{3}a-3\frac{2}{3}b$, $-a-3b$, $-a-3b$, &c.
24. 5. **25.** 6. **26.** 101. **27.** -25.

LXIV. **1.** 420. **2.** 180. **3.** $94\frac{1}{2}$. **4.** 660.
5. 0. **6.** $-2104\frac{1}{2}$. **7.** 63. **8.** 0.
9. $\frac{1}{12}(45n-5n^2)$. **10.** $\frac{1}{21}(7n^2+17n)$. **11.** 0.
12. 0. **13.** 357. **14.** $7\sqrt{2}+14$. **15.** $\frac{1}{2}(n-1)$.
16. $\frac{1}{2}(n^3+3n^2)$. **17.** $n(a+b)^2-n(n-1)ab$. **18.** 375.
19. 1878. **20.** 880. **21.** 2400.
22. $6\frac{1}{2}$, 8 &c.; sum is $797\frac{1}{2}$.
23. $12\frac{8}{41}$, $14\frac{16}{41}$, &c.; sum is 2200. **24.** 55, 1485.
25. 511, 36281. **26.** 38. **27.** 680. **28.** 2415.
29. 3. **30.** $-\frac{5}{12}$. **31.** 1. **32.** $\frac{1}{2}$. **33.** 48.
34. 5. **35.** 9. **36.** 5 or 6.
37. $3a$ or $-2a$ provided either of them is a positive integer.
38. 21. **39.** 9. **40.** 3. **41.** 4060. **42.** 90.
43. 525. **44.** $200n+100$. **47.** 7500. **48.** 246950.
49. 39900. **50.** 17. **51.** 1, 4, 7, &c.
53. 20, 21, 22, &c. **54.** 43. **55.** 10.
56. 8, 16, 24, 32. **57.** ± 2, ± 4, ± 6, ± 8.

LXV. **1.** (i) $\dfrac{1}{27}$, (ii) $-\dfrac{243}{16}$, (iii) $\dfrac{b^5}{a^3}$. **2.** $\dfrac{16}{3}$.
3. $-\frac{1}{128}$. **4.** 2. **5.** ± 1092. **6.** 5·0625.
7. $-\frac{1}{32}$. **8.** $\frac{1}{162}$. **9.** (i) ± 6, (ii) ± 42, (iii) $\pm a^2b^2$.
10. -2, 4; 6, 3, $\frac{3}{2}$; 1, $\frac{3}{2}$, $\frac{9}{4}$, $\frac{27}{8}$.
11. ·6, ·12, ·024, ·0048, ·00096. **12.** a^2b^{-4}, ab^{-2}, 1, $a^{-1}b^2$.
13. $\frac{1}{2}$. **14.** 54 or 108.

LXVI. **1.** $16\left\{\left(\dfrac{3}{2}\right)^n-1\right\}$. **2.** $48\left\{1-\left(\dfrac{3}{4}\right)^6\right\}$.
3. $\dfrac{a(r^n-1)}{r^{n-1}(r-1)}$. **4.** $\dfrac{27}{2}\left\{1-\left(\dfrac{8}{9}\right)^8\right\}$. **5.** $\dfrac{4}{9}\left\{\left(\dfrac{3}{2}\right)^6-1\right\}$.

6. $\frac{48}{7}$. 7. $\frac{40}{7}$. 8. 5. 9. $\frac{a^n-b^n}{a^{n-3}(a-b)}$.
10. $\frac{9}{5}$. 11. 8190. 12. 3·4142. 13. $6x^2-5x-6$.
15. $1, \frac{3}{4}, \frac{9}{16}$, &c. or $3, \frac{3}{4}, \frac{3}{16}$, &c. 16. $12, -4, \frac{4}{3}$, &c.
17. 2. 20. 8, 16, 32,... 21. $\frac{a}{b}$. 22. a.
24. $\frac{9}{46}(3\sqrt{6}+2\sqrt{2})$. 25. $\frac{7}{12}$ or $-\frac{7}{16}$. 26. 2, 4, 8.
27. 2, 8, 32,.... 28. $\frac{1}{2}, \pm 1, 2, \ldots$.

LXVII. 1. 0. 2. $\frac{1}{4}(9^7-1)$. 3. $\frac{5}{4}$. 4. $\frac{b^n-a^n}{a^{n-2}(b-a)}$.
5. $\frac{1}{3}$. 6. $104\frac{1}{2}$. 7. $76a+57b$. 8. $\frac{3}{28}\{1-(\frac{4}{3})^8\}$.
9. $5\frac{5}{24}$. 10. $\frac{5}{3}\{(\frac{5}{2})^5-1\}$. 11. (i) $71\frac{1}{4}$. (ii) -6.
(iii) $\frac{16}{7}\{1-(\frac{3}{4})^6\}, \frac{16}{7}$. (iv) $\frac{2}{3}\{(\frac{3}{2})^6-1\}$. (v) $\frac{3}{2}\{1-(\frac{2}{3})^6\}, \frac{3}{2}$.
13. 1, 3, 5,.... 14. $5\pm\sqrt{24}$. 17. 2, 6, 18.
18. 1, 3, 5, 7, 9. 19. 6, 9, 12, 16 or 6, 1, -4, 16.

LXVIII. 1. $\frac{7}{6}, \frac{7}{5}, \frac{7}{4}, \frac{7}{3}, \frac{7}{2}$. 2. $\frac{3}{4}, \frac{6}{7}, 1, \frac{6}{5}$. 20. $6\frac{2}{3}, 5\frac{1}{3}, 4$.

LXIX. 1. $\frac{4}{3}n(n+1)(n+2)$.
2. $\frac{1}{6}\{(2n+1)(2n+3)(2n+5)-15\}$.
3. $\frac{1}{9}\{(3n-2)(3n+1)(3n+4)+8\}$.
4. $\frac{1}{9}\{(3n-1)(3n+2)(3n+5)+10\}$.
5. $\frac{1}{8}\{(2n-1)(2n+1)(2n+3)(2n+5)+15\}$.
6. $\frac{1}{20}\{(5n-3)(5n+2)(5n+7)(5n+12)+3\,.\,2\,.\,7\,.\,12\}$.
7. $\frac{1}{2}-\frac{1}{2(2n+1)}, \frac{1}{2}$. 8. $\frac{1}{6}-\frac{1}{3(3n+2)}, \frac{1}{6}$.
9. $\frac{1}{12}-\frac{1}{4(2n+1)(2n+3)}, \frac{1}{12}$.
10. $\frac{1}{60}-\frac{1}{6(3n+2)(3n+5)}, \frac{1}{60}$.
11. $\frac{1}{x+1}-\frac{1}{x+n+1}, \frac{1}{x+1}$.
12. $\frac{1}{x}\left\{\frac{1}{1+x}-\frac{1}{1+(n+1)x}\right\}, \frac{1}{x(x+1)}$.
13. $2\left(\frac{1}{1}-\frac{1}{n+1}\right)$, 2. 14. $\frac{1}{3}n(4n^2-1)$. 15. $n^2(2n^2-1)$.
16. $\frac{1}{3b}\{(a+\overline{n-1}b)(a+nb)(a+\overline{n+1}b)-(a-b)a(a+b)\}$.

MISCELLANEOUS EXAMPLES. VI.

A. 1. 48. 3. $3x-7$, $x=\frac{7}{3}$.
4. (i) $x=-4$, $y=\frac{1}{7}$, (ii) $x=c$, or $x=a+b+c$.
5. £20 each ox, £2. 10*s*. each sheep.
7. $ab^{-1}+32a^{-\frac{2}{3}}b^{\frac{3}{2}}$. 8. $-\frac{1}{23}(73+40\sqrt{3})$, $\sqrt{8}+\sqrt{17}$.
9. $\frac{20}{7}\{1-(\frac{3}{4})^8\}$, $\frac{20}{7}$. 10. $109\frac{11}{21}$, $119\frac{1}{21}$, ... $290\frac{10}{21}$, sum 4000.

B. 1. $x-8y+4z$. 2. $(nx+my)(mx+ny)$, $(x-y)(x^2+y^2)$.
4. (i) $x=\pm\sqrt{ab}$, (ii) $x=7$ or -2. 6. a^3+b^2-c.
8. (i) -1040, (ii) $4\frac{1}{6}$. 9. 71071. 10. 88 yards.

C. 1. 2. 2. $2x^2-3xy+2y^2$. 4. $3(x-3a)^2(x^2-4a^2)$.
5. $\dfrac{6a^2}{(a^2-4)(a^2-1)}$. 6. 25 miles an hour, 100 miles.
7. 1. 9. (i) 0, (ii) $8\frac{1}{3}$. 10. 9 and 25.

D. 1. x^4-a^4. 3. $\dfrac{(x-4)(x-7)}{x^2}$.
4. (i) $x=\frac{1}{2}$, $y=-\frac{1}{3}$, (ii) $x=\pm5$, $y=\pm3$; $x=\pm\frac{13}{3}$, $y=\pm\frac{5}{3}$.
5. 48. 6. $x^2-40x+8=0$. 7. $x^3-x^{-1}+4x^{-3}$.
9. (i) $-7\frac{853}{1536}$, (ii) $5\frac{2}{5}$.

E. 1. $-x^2-xy-y^2$. 2. $3a^2+4ab+b^2$.
3. $a-b$, $(a-b)(4a-b)(3a^2+b^2)$. 4. $(x+1)^2$. 5. $-mn$.
6. $x=-\frac{5}{6}$. 8. $2x^{\frac{2}{3}}-3x^{\frac{1}{3}}y^{\frac{1}{2}}-4y$. 9. (i) $\frac{1}{2}n(n-7)$, (ii) $\frac{2}{3}$.
10. $22\frac{1}{2}$ and 45.

F. 1. $8xz$. 2. $a^2+b^2+c^2-2ab+ac-bc$. 4. $-\dfrac{2}{a+2x}$.
5. (i) $x=3$, $y=7$, $z=4$. (ii) $x=4\frac{1}{2}$. (iii) $x=\pm9$, $y=\pm3$.
6. Expression is $(x^3-2x^2+5x-6)^2$. 7. $\frac{1}{2}(\sqrt{5}+\sqrt{7})=2{\cdot}4409$.

LXX. 1. 2730. 2. 5040. 3. 40320. 4. 120.
5. 1956. 6. $8\,\underline{|9}$. 7. 6. 8. 420, 34650, $\frac{1}{2}\,\underline{|9}$.
9. 1260. 10. 210. 11. 1120, 180. 12. 720.
13. 8. 14. 1820, 455, 190. 15. 105. 16. 1330.

17. 48. **18.** 5. **19.** 5. **20.** 90. **21.** 125.

22. 59. **23.** 209. **28.** 210. **29.** 25.

30. 246480. **31.** $\dfrac{|\underline{30}}{|\underline{11}\,|\underline{19}}$, $\dfrac{1}{2}\dfrac{|\underline{30}}{|\underline{11}\,|\underline{11}\,|\underline{8}}$. **32.** 60. **33.** 180.

34. $|\underline{7}$. **35.** $\frac{1}{2}n(n-1)$, $\frac{1}{48}n(n+1)(n-1)(n-2)(n^2-n-4)$.

LXXI. **1.** $x^6+6ax^5+15a^2x^4+20a^3x^3+15a^4x^2+6a^5x+a^6$.

2. $1-5x^2+10x^4-10x^6+5x^8-x^{10}$.

3. $81x^4-216x^3y+216x^2y^2-96xy^3+16y^4$.

4. $16a^4+96a^5+216a^6+216a^7+81a^8$.

5. $64x^{12}-192x^{10}+240x^8-160x^6+60x^4-12x^2+1$.

6. $y^7-7xy^6+21x^2y^5-35x^3y^4+35x^4y^3-21x^5y^2+7x^6y-x^7$.

7. $405a^8b^2$. **8.** $126720x^{16}$. **9.** $-14a^3$.

10. $210x^6$. **11.** $1140x^{17}$. **12.** $231x^{20}$.

13. $2a^4+12a^2b+2b^2$. **14.** $2a^6+30a^4b+30a^2b^2+2b^3$.

15. $2a^8+56a^6b+140a^4b^2+56a^2b^3+2b^4$.

16. $70x^4$. **17.** $252x^5$. **18.** $90720x^4y^4$.

20. $x^{16}+8x^{12}+28x^8+56x^4+70+56x^{-4}+28x^{-8}+8x^{-12}+x^{-16}$.

21. 3003. **22.** $\dfrac{|\underline{20}}{|\underline{16}\,|\underline{4}}\dfrac{1}{x^8}$. **23.** $(-1)^n\cdot\dfrac{|\underline{2n}}{|\underline{n}\,|\underline{n}}a^{3n}$.

24. $\dfrac{|\underline{2n+1}}{|\underline{n}\,|\underline{n+1}}x$, $\dfrac{|\underline{2n+1}}{|\underline{n}\,|\underline{n+1}}\dfrac{1}{x}$. **25.** The fifth.

26. The fifth and sixth. **27.** The ninth.

28. The eleventh. **29.** The sixth.

30. The fourteenth and fifteenth.

31. $(-1)^{n-r}\dfrac{|\underline{n}}{|\underline{r}\,|\underline{n-r}}a^rb^{n-r}$. **34.** 96059601, 997002999.

LXXIII. **1.** $1+4x+10x^2+20x^3+35x^4+\ldots$.

2. $1-8x+40x^2-160x^3+560x^4-\ldots$.

3. $\frac{1}{8}\left\{1+\frac{3}{2}x+\frac{3}{2}x^2+\frac{5}{4}x^3+\frac{15}{16}x^4+\frac{21}{32}x^5+\ldots\right\}$.

4. $1+2x+5x^2+\frac{40}{3}x^3+\frac{110}{3}x^4+\ldots$.

5. $1 - 3x - 3x^2 - 7x^3 - 21x^4 - \ldots$.

6. $\dfrac{(r+1)(r+2)(r+3)(r+4)}{\lfloor 4} x^r$. 7. $\dfrac{n(n+1)\ldots(n+r-1)}{\lfloor r} x^r$.

8. $\dfrac{3 . 5 . 7 \ldots (2r+1)}{2^r \lfloor r} x^r$. 9. $(-1)^r \dfrac{3 . 1 . 3 . 5 \ldots (2r-5)}{2^r \lfloor r} x^r$.

10. $(-1)^r \dfrac{2 . 5 . 8 \ldots (3r-1)}{3^r \lfloor r} x^r$. 11. $\dfrac{5 . 7 . 9 \ldots (2r+3)}{\lfloor r} x^r$.

12. $(-1)^r \dfrac{4 . 7 . 10 \ldots (3r+1)}{\lfloor r} x^r$. 13. $\dfrac{5}{16a^6}$.

14. $1 - 2x - x^2 - \dfrac{4}{3} x^3 - \dfrac{7}{3} x^4 - \ldots;\ -2 \dfrac{1 . 4 . 7 \ldots (3r-5)}{\lfloor r} x^r$.

15. -4; -11. 16. $-\frac{31}{16}$, $\frac{107}{128}$. 17. $-\frac{1701}{256} x^5$.

18. $-\frac{1729}{768} x^5$. 19. 10·4880…, 5·0657…, 5·0099….

20. 10, 3.

LXXIV. 1. 0. 2. 4. 3. -2. 4. -3.

5. -1. 6. $\frac{2}{5}$. 7. 5. 8. $\frac{5}{2}$. 9. 2.

10. $\frac{7}{4}$. 11. $\frac{3}{4}$. 12. $\frac{3}{7}$.

13. $-$·1760913, 1·7781513, 3·6532126, ·2795880.

14. ·5740313, ·1072100, 1·4210639.

15. 2·5105452, ·1637578, $-$2·3645161.

16. ·0969100, ·1072100, ·7886320.

17. ·9030900, ·9542426. 18. 1·8573326, ·5270968.

LXXV. 1. 200000, ·00002. 2. 1991·1, ·00019911.

3. $\bar{2}$·6707522. 4. $\bar{3}$·7705677. 5. 1·961109.

6. 1·93070. 7. 1·71877. 8. ·324838.

9. £131. 10*s*. 11. £742. 19*s*. 6*d*.

12. £641. 17*s*. 3*d*.

LXXVI. 1. $(x+1)(x-1)^2$. 2. $(x^2+1)(x+1)^2(x-1)^2$.

3. $(1-x)^2(1+x)(1+x^2)$.

4. $(1-x)^2(1+x)^2(1+x^2)(1+x^4)$.

5. $(x-y-a-b)(x+y-a+b)$. 6. $(x+y-1)(x-y-2)$.

7. $(x+2y-4)(x-4y+2)$. 8. $(a+b-c+f)(a-b-c-f)$.

9. (i) $x^4+x^3y+x^2y^2+xy^3+y^4$.

(ii) $x^6-x^5y+x\ y^2-x^3y^3+x^2y^4-xy^5+y^6$.

(iii) $x^7-x^6y+x^5y^2-x^4y^3+x^3y^4-x^2y^5+xy^6-y^7$.

12. $(2a+3b)^2-(2a+3b)(3a+2b)+(3a+2b)^2$.

13. $3(2a+4b-4c)^2-3(2a+4b-4c)(a-b+7c)+3(a-b+7c)^2$.

LXXVII. 9. $-(b-c)(c-a)(a-b)(a+b+c)$. 10. $4abc$.

11. $2abc$. 12. $2abc(a+b+c)$.

13. 3 ; $5(x^2+y^2+z^2+yz+zx+xy)$. 15. $a+b+c$.

16. 1. 17. $\dfrac{1}{abc}$. 18. $\dfrac{bc+ca+ab}{a^2b^2c^2}$. 19. 0.

20. $\dfrac{a+b+c}{abc}$. 24. $(b+c)(c+a)(a+b)$.

45. (i) $x-8y$. (ii) $1+x+x^2$. (iii) x^2-3x+2. (iv) $2x^2-3x+4$.

LXXVIII. 1. 13552, 53221, 4043001.

2. 761, 1477, 7212, 22875. 3. 140e, 9222.

4. $43\cdot 20\dot{2}$, $443\cdot 2\dot{1}$. 5. $3\cdot\dot{4}$. 7. $a=b=1$.

8. $a=9$, $b=6$. 9. 7. 10. 13. 11. 25.

12. 15.

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